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Vol-Bond: an analytical solution

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Abstract

We find an analytical solution of the Vol-Bond according to the multi-factor Gaussian Heath–Jarrow–Morton model. We show how to calibrate the model with market data. This solution allows us to have complete (and fast) control of this class of derivatives and of their sensitivities.

Fixed-income exotic derivatives are becoming the new frontier in derivative pricing. They allow one to tailor contracts to the client's real exigencies and often to design products whose market risk can be easily managed. Unfortunately, even in relatively simple cases no analytical solution is available in the literature. In this paper we show how to solve analytically a particular exotic product, the Vol-Bond, according to the multi-factor Gaussian Heath–Jarrow–Morton (HJM) model [1].

A Vol-Bond is a straddle, where at the reset date T_i one compares the Libor rate which fixes at time T_i with Libor's fixing at time T_{i-1}. At date T_{i+1} the pay-off of a Vol-Bond (with N reset dates and last payment date T_{N+1}) is

$$\theta |L(T_i; T_i, T_{i+1}) - L(T_{i-1}; T_{i-1}, T_i)| \qquad i = 1, \ldots, N, \tag{1}$$

where the strike resets at the Libor previous fixing (T_{i-1}), the pay-off is fixed up-front (T_i) and paid in arrears (T_{i+1}) and $L(T_a; T_b, T_c)$ is the Libor rate between T_b and T_c considered at time T_a with $T_a \leqq T_b < T_c$. For simplicity we deal with the case in which each reset date lags θ after the previous one.

Using the standard notation, the following relations hold (see e.g. [2–4]):

$$L(T_a; T_b; T_c) = \frac{1}{\theta}\left(\frac{1}{B(T_a; T_b, T_c)} - 1\right) \tag{2}$$

where $B(T_a; T_b, T_c)$ is the value in T_a of the zero coupon starting in T_b and ending in T_c and

$$B(T_a; T_a, T_b) \equiv B(T_a, T_b) \equiv E\left[\mathrm{e}^{-\int_{T_a}^{T_b} r_t\,\mathrm{d}t} | \mathcal{F}_{T_a}\right] = \mathrm{e}^{-\int_{T_a}^{T_b} f(T_a, t)\,\mathrm{d}t} \tag{3}$$

where r_t is the spot rate and $f(T_a, t)$ is the instantaneous forward rate between T_a and t.

The value of a Vol-Bond can be seen as a linear combination of caplets and floorlets:

$$\mathrm{VB} \equiv \sum_{i=1}^{N} (\mathcal{C}_i + \mathcal{F}_i) \tag{4}$$

where the ith caplet is

$$\mathcal{C}_i \equiv \theta E[\mathrm{e}^{-\int_0^{T_{i+1}} r_t\,\mathrm{d}t} (L(T_i; T_i, T_{i+1}) - L(T_{i-1}; T_{i-1}, T_i))^+] \tag{5}$$

and the ith floorlet is

$$\mathcal{F}_i \equiv \theta E[\mathrm{e}^{-\int_0^{T_{i+1}} r_t\,\mathrm{d}t} (L(T_{i-1}; T_{i-1}, T_i) - L(T_i; T_i, T_{i+1}))^+]. \tag{6}$$

To evaluate equations (5) and (6) we use the multi-factor Gaussian HJM model (MHJM in the following). The HJM model on one hand satisfies the no-arbitrage condition and on the other allows calibration to the initially observed term structure of zero coupons $B(0, T)$. Furthermore, in the MHJM as in the BGM model [5] we can have an evolution of the zero-coupon term structure according to several degrees of freedom. However, while in the BGM model only an approximated solution is available, e.g. using the drift-freezing technique (see [3, 6])[1], in the MHJM it is possible to solve the problem analytically and this allows one to show explicitly the impact of the different degrees of freedom on the rates–term–structure dynamics.

The MHJM model assumes that, under the risk-neutral measure, the dynamics for the instantaneous forward rate is

$$\mathrm{d}f(t, T) = \frac{1}{2}\frac{\partial}{\partial T}\sigma(t, T)\rho\sigma(t, T)\,\mathrm{d}t - \frac{\partial}{\partial T}\sigma(t, T)\,\mathrm{d}W_t \tag{7}$$

where $\sigma(t, T)$ is an M-dimensional deterministic function of time with $\sigma(T, T) = 0$ and W is an M-dimensional Brownian

[1] We thank a referee for sorting out this point.

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motion with instantaneous covariance $\rho = (\rho_{i,j=1,\dots,M})$:

$$\mathrm{d}W_{t,i}\,\mathrm{d}W_{t,j} = \rho_{i,j}\,\mathrm{d}t.$$

The product $\sigma(t,T)\,\mathrm{d}W_t$ is the scalar product of the two vectors, $\sigma(t,T)$ and $\mathrm{d}W_t$.

Equation (7) is equivalent to considering (see e.g. [8])

$$\mathrm{d}B(t,T) = B(t,T)[r_t\,\mathrm{d}t + \sigma(t,T)\,\mathrm{d}W_t] \tag{8}$$

and

$$r_t = f(0,t) + \frac{1}{2}\int_0^t \frac{\partial}{\partial t}\sigma(t',t)\rho\sigma(t',t)\,\mathrm{d}t' - \int_0^t \frac{\partial}{\partial t}\sigma(t',t)\,\mathrm{d}W_{t'}. \tag{9}$$

The MHJM model is of particular interest since an analytical solution for caplet and floorlet is available in the plain vanilla case [2, 7] and then it allows one to calibrate the model to market volatilities.

The plain vanilla caplet with strike K is equal to

$$\begin{aligned} C_i &\equiv \theta E[\mathrm{e}^{-\int_0^{T_{i+1}} r_t\,\mathrm{d}t}(L(T_i;T_i,T_{i+1}) - K)^+] \\ &= B(0;T_{i+1})[(1+\theta L(0;T_i,T_{i+1}))N(d_1^{(PV)}) \\ &\quad - (1+\theta K)N(d_2^{(PV)})], \end{aligned} \tag{10}$$

where

$$d_1^{(PV)} = \frac{1}{\mathcal{V}_i}\ln\frac{1+\theta L(0;T_i,T_{i+1})}{1+\theta K} + \frac{1}{2}\mathcal{V}_i,$$

$$d_2^{(PV)} = \frac{1}{\mathcal{V}_i}\ln\frac{1+\theta L(0;T_i,T_{i+1})}{1+\theta K} - \frac{1}{2}\mathcal{V}_i$$

and

$$\mathcal{V}_i^2 \equiv \int_0^{T_i} [\sigma(t,T_{i+1}) - \sigma(t,T_i)]\rho[\sigma(t,T_{i+1}) - \sigma(t,T_i)]\,\mathrm{d}t. \tag{11}$$

Similarly the ith floorlet with strike K is

$$\begin{aligned} F_i &\equiv \theta E[\mathrm{e}^{-\int_0^{T_{i+1}} r_t\,\mathrm{d}t}(K - L(T_i;T_i,T_{i+1}))^+] \\ &= B(0;T_{i+1})[(1+\theta K)N(-d_2^{(PV)}) \\ &\quad - (1+\theta L(0;T_i,T_{i+1}))N(-d_1^{(PV)})]. \end{aligned} \tag{12}$$

Theorem. *Under the hypothesis of the MHJM model, the Vol-Bond is*

$$\mathrm{VB} \equiv \sum_{i=1}^{N}(\mathcal{C}_i + \mathcal{F}_i) \tag{13}$$

where the ith caplet of a Vol-Bond is equal to

$$\begin{aligned} \mathcal{C}_i &= B(0,T_{i+1})[(1+\theta L(0;T_i,T_{i+1}))N(d_1) \\ &\quad - \mathcal{K}(1+\theta L(0;T_{i-1},T_i))N(d_2)] \end{aligned} \tag{14}$$

and the ith floorlet is

$$\begin{aligned} \mathcal{F}_i &= B(0;T_{i+1})[\mathcal{K}(1+\theta L(0;T_{i-1},T_i))N(-d_2) \\ &\quad - (1+\theta L(0;T_i,T_{i+1}))N(-d_1)] \end{aligned} \tag{15}$$

with

$$d_1 \equiv \frac{1}{\mathcal{G}_i}\ln\frac{1+\theta L(0;T_i,T_{i+1})}{1+\theta L(0;T_{i-1},T_i)}\frac{1}{\mathcal{K}} + \frac{1}{2}\mathcal{G}_i$$

$$d_2 \equiv \frac{1}{\mathcal{G}_i}\ln\frac{1+\theta L(0;T_i,T_{i+1})}{1+\theta L(0;T_{i-1},T_i)}\frac{1}{\mathcal{K}} - \frac{1}{2}\mathcal{G}_i.$$

We define

$$\mathcal{G}_i^2 \equiv \int_0^{T_i} g_i(t)\rho g_i(t)\,\mathrm{d}t \tag{16}$$

with

$$g_i(t) \equiv \begin{cases} (\sigma(t,T_{i+1}) - \sigma(t,T_i)) - (\sigma(t,T_i) - \sigma(t,T_{i-1})) & t \text{ in } [0,T_{i-1}) \\ (\sigma(t,T_{i+1}) - \sigma(t,T_i)) & t \text{ in } [T_{i-1},T_i] \end{cases}$$

and

$$\mathcal{K} \equiv \exp\left(-\int_0^{T_{i-1}} (\sigma(t,T_{i+1}) - \sigma(t,T_i))\rho(\sigma(t,T_i) - \sigma(t,T_{i-1}))\,\mathrm{d}t\right). \tag{17}$$

Proof. Let us consider the caplet case (*mutatis mutandis*, the proof is the same for a floorlet). Recalling the relation (2) between Libor rates and zero coupons, equation (5) is equivalent to

$$\mathcal{C}_i = E\left[\mathrm{e}^{-\int_0^{T_i} r_t\,\mathrm{d}t}\left(1 - \frac{B(T_i;T_i,T_{i+1})}{B(T_{i-1};T_{i-1},T_i)}\right)^+\right].$$

The above equation, applying the change-of-numeraire technique [9], can be rewritten as

$$\mathcal{C}_i = B(0;T_i)E^{(T_i)}\left[1 - \frac{B(T_i;T_i,T_{i+1})}{B(T_{i-1};T_{i-1},T_i)}\right]^+ \tag{18}$$

where $E^{(T_i)}[\cdot]$ is the expectation under the forward measure $Q^{(T_i)}$ and

$$W_t^{(T_i)} \equiv W_t - \int_0^{T_i} \rho\sigma(t,T_i)\,\mathrm{d}t$$

is the martingale under the forward measure $Q^{(T_i)}$.

After some algebra, it is straightforward to show that

$$\frac{B(T_i;T_i,T_{i+1})}{B(T_{i-1};T_{i-1},T_i)} = \frac{B(0;T_i,T_{i+1})}{B(0;T_{i-1},T_i)} \times \mathcal{K}\exp\left(-\tfrac{1}{2}\int_0^{T_i} g_i(t)\rho g_i(t)\,\mathrm{d}t + \int_0^{T_i} g_i(t)\,\mathrm{d}W_t^{(T_i)}\right).$$

The theorem is proven on applying the Girsanov transform

$$W_t^{\prime(T_i)} = W_t^{(T_i)} - \int_0^{T_i} \rho g_i(t)\,\mathrm{d}t$$

to the second integral of equation (18). □

Let us briefly comment the result. Let us consider an N-factor model where only the ith component of the volatility $(\sigma(t,T_{i+1}) - \sigma(t,T_i))$ is different from zero, i.e. $(\sigma(t,T_{i+1}) - \sigma(t,T_i))_l = v_i(t)\delta_{i,l}$ where $v_i(t)$ is one-dimensional function of time and $\delta_{i,l}$ is the Dirichlet delta.

In this case $v_i(t)$ can be calibrated directly on market data, using equations (10) and (12) and observing that equation (11) can be rewritten as

$$\mathcal{V}_i^2 \equiv \int_0^{T_i} v_i^2(t)\,\mathrm{d}t \tag{19}$$

and

$$\rho_{i-1,i} = \mathrm{corr}(\mathrm{d}\ln B(t; T_i, T_{i+1}), \mathrm{d}\ln B(t; T_{i-1}, T_i))$$

and then, due to relation (2), $\rho_{i-1,i}$ depends only on two consecutive Libor rates:

$$\rho_{i-1,i} = \mathrm{corr}(\mathrm{d}\ln(1+\theta L(t; T_i, T_{i+1})), \times \mathrm{d}\ln(1+\theta L(t; T_{i-1}, T_i))).$$

We can rewrite in equation (16)

$$g_i(t)\rho g_i(t) = \begin{cases} v_{i-1}(t)^2 - 2\rho_{i-1,i}v_{i-1}(t)v_i(t) + v_i(t)^2 & t \text{ in } [0, T_{i-1}) \\ v_i(t)^2 & t \text{ in } [T_{i-1}, T_i] \end{cases}$$

and equation (17) as

$$\mathcal{K} = \exp\left(-\int_0^{T_{i-1}} \rho_{i-1,i} v_{i-1}(t)v_i(t)\,\mathrm{d}t\right)$$

and then the ith Vol-Bond caplet and floorlet depend only on $\rho_{i-1,i}$.

We notice that equation (14) is similar to a plain vanilla caplet (see equation (10)) with an 'effective' strike $\mathcal{K}$ which modifies only slightly the option moneyness. Furthermore, when the correlation $\rho_{i-1,i}$ is not so different from 1 (generally for $T_i > 1$ year) [3], we observe the similarity with a forward start option in equity derivatives: in the caplet $\mathcal{C}_i$ and in the floorlet $\mathcal{F}_i$ the diffusion term is negligible up to T_{i-1}, since

$$v_{i-1}(t)^2 - 2\rho_{i-1,i}v_{i-1}(t)v_i(t) + v_i(t)^2 \ll v_i(t)^2.$$

This allows one to understand why a Vol-Bond is mainly a derivative product of volatility, since it depends only on the correlation between two functions of consecutive forward Libor rates. It also clarifies why the Vol-Bond can be significantly cheaper than the plain vanilla ATM straddle.

In this paper we have derived an analytical solution for a Vol-Bond in the MHJM framework; we have also shown how to calibrate the model with market data (rates and Vols-term structures and correlations). An analytical approach, besides the (obvious) advantage of a fast computation of derivative price and of its greeks, can provide the basic brick for building up more exotic pay-offs.

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