A simple solution for
Sticky Cap and Sticky Floor

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Abstract

We show an analytical approach to Sticky Cap and Sticky Floor according to the Multi-Factor Gaussian Heath-Jarrow-Morton model and a calibration procedure is described for all the parameters involved in the model. This solution allows having a comprehensive approach even for this class of Fixed Income exotic derivatives that are fully path dependent.

Why is it so tough to price and manage risks involving Fixed Income exotic derivatives? One reason is surely due to the fact that these contingent claims involve the several degrees of freedom in the forward rates curve and that in the literature very few closed form solutions are available. In this letter we show how to price sticky caps and floors, FI exotic derivatives well known in the market (see e.g. [1, 2]) that, due to its path dependence, fully involve the different features of a multi-factor interest rate model. Another reason is that, even when a solution is available, a calibration for model parameters is not straightforward. We show how to calibrate all the parameters involved in the multi-factor model considered.

In a sticky cap (as in a plain vanilla cap) there are $N$ possible payments. The $n^{th}$ cap rate $K_n$ established at time $T_n$ is the minor between Libor rate with fixing at time $T_n$ and previous rate $K_{n-1}$. Likely in a sticky floor, $K_n$ is the maximum between Libor rate with fixing at time $T_n$ and previous rate $K_{n-1}$. Each payment in this exotic derivative depends then on all the previous payments: it is completely path dependent. In this letter we show that, according to the Multifactor Gaussian Heath-Jarrow-Morton [3] Model (MHJM in the following) in the form introduced in [4], it is possible to derive an analytical solution for a sticky cap and a sticky floor.

Following the standard notation, let us define $L_n(T_n)$ as the Libor rate between $T_n$ and $T_{n+1}$ and fixing in $T_n$; for simplicity we deal with the case in which each reset date is a lag $\theta$ after the previous one.

The $n^{th}$ sticky cap payoff reads:

$$\theta \min (L_n(T_n), K_{n-1}) \quad n = 1, \ldots, N$$
and the \( n^{th} \) sticky floor payoff is:

\[
\theta \max (L_n(T_n), K_{n-1}) \quad n = 1, \ldots, N
\]

where the \( n^{th} \) payoff is established in \( T_n \), calculated for the lag \( \theta = T_{n+1} - T_n \) and paid in \( T_{n+1} \).

At time \( T_0 = 0 \) the value of a sticky cap is

\[
C \equiv \sum_{n=1}^{N} \theta E \left[ e^{-\int_{0}^{T_{n+1}} r_t dt} \min_{j=0, \ldots, n} L_j(T_j) \right]
\]

(1)

and a stick floor

\[
\mathcal{F} \equiv \sum_{n=1}^{N} \theta E \left[ e^{-\int_{0}^{T_{n+1}} r_t dt} \max_{j=0, \ldots, n} L_j(T_j) \right]
\]

(2)

where \( r_t \) is the spot rate.

To evaluate eq. (1) and eq. (2) we use the MHJM Model. This model assumes that, under the risk-neutral measure, the dynamics for the instantaneous forward rate \( f(t, T) \) between \( t \) and \( T \) is

\[
df(t, T) = \frac{1}{2} \frac{\partial}{\partial T} \sigma(t, T) \rho \sigma(t, T) dt - \frac{\partial}{\partial T} \sigma(t, T) dW_t
\]

(3)

where \( \sigma(t, T) \) is an \( M \)-dimensional deterministic function of time with \( \sigma(T, T) = 0 \) and \( W \) is an \( M \)-dimensional Brownian motion with instantaneous covariance \( \rho = (\rho_{i,j=1,\ldots,M}) \)

\[
dW_{t,i}dW_{t,j} = \rho_{i,j} dt
\]

and \( \sigma(t, T)dW_t \) is the scalar product between the two vectors \( \sigma(t, T) \) and \( dW_t \).

Equation (3) is equivalent to (see e.g. [5])

\[
dB(t, T) = B(t, T)[r_t dt + \sigma(t, T)dW_t]
\]

(4)

where \( B(t, T) \) is the value in \( t \) of the zero coupon which pays 1 in \( T \) and the instantaneous rate

\[
r_t = f(0, t) + \frac{1}{2} \int_{0}^{t} \frac{\partial}{\partial t'\partial t} \sigma(t', t) \rho \sigma(t', t) dt' - \int_{0}^{t} \frac{\partial}{\partial t} \sigma(t', t) dW_{t'}
\]

(5)
Defining $L_i(t)$ the Forward Rate Agreement in $t$ of the Libor Rate $L_i(T_i)$, the following relations hold (see e.g. [6, 7, 2]):

$$L_i(t) = \frac{1}{\theta} \left( \frac{1}{B_i(t)} - 1 \right)$$

and we define $B_i(t) = B(t, T_{i+1})/B(t, T_i)$ and $\nu_i(t) = \sigma(t, T_{i+1}) - \sigma(t, T_i)$.

**THEOREM**: Under the hypothesis of the Multi-Factor Gaussian HJM Model, the sticky cap is equal to

$$C \equiv \sum_{n=1}^{N} B(0, T_{n+1}) \left\{ \sum_{j=0}^{n} [1 + \theta L_j(0)] K^{(n,j)} N_n(d^{(n,j)}, R^{(n,j)}) - 1 \right\}$$

and the sticky floor is

$$F \equiv \sum_{n=1}^{N} B(0, T_{n+1}) \left\{ \sum_{j=0}^{n} [1 + \theta L_j(0)] K^{(n,j)} N_n(-d^{(n,j)}, R^{(n,j)}) - 1 \right\}$$

where $B(0, T)$ is the zero coupon starting at $T_0 = 0$ and ending in $T$, $N_n(d, R)$ stands for the $n$-variate normal distribution evaluated at the vector $d$ and with correlation matrix $R$ and

$$K^{(n,j)} \equiv \begin{cases} \text{Exp} \left[ - \int_0^{T_j} v_j(t) \rho \sum_{l=j+1}^{n} \nu_l(t) dt \right] & 0 \leq j < n \\ 1 & j = n \end{cases}$$

$$d^{(n,j)}_i \equiv \begin{cases} \frac{1}{\Sigma^{(n,j)}_i \sqrt{T_n}} \ln \left( \frac{B_i(0)K^{(n,j)}}{B_j(0)K^{(n,i)}} \right) - \frac{1}{2} \Sigma^{(n,j)}_i \sqrt{T_n} & 0 \leq j < n \ i \neq j \ i = 0, \ldots, n-1 \\ \frac{1}{G^{(n)}_j \sqrt{T_n}} \ln \left( \frac{B_j(0)K^{(n,j)}}{B_i(0)K^{(n,i)}} \right) - \frac{1}{2} G^{(n)}_j \sqrt{T_n} & 0 \leq j < n \ i = j \end{cases}$$

$$R^{(n,j)}_{il} \equiv \begin{cases} \frac{G^{(n)}_j - \eta^{(n)}_{ij} G^{(n)}_i}{\Sigma^{(n,j)}_i G^{(n)}_i} & 0 \leq j < n \ i, l \neq j \ i, l < n \\ \frac{1}{\eta^{(n)}_{ii}} & j = n \end{cases}$$
Proof: Let us consider the cap case since *mutatis mutandis* the proof is the same for the floor. Recalling the relation (6) between Libor Rates and Zero Coupons, equation (1) is equivalent to

\[
C = \sum_{n=1}^{N} \left\{ B(0, T_n) \mathbb{E}^{(T_n)} \left[ \min_{j=0, \ldots, n} \frac{B_n(T_n)}{B_j(T_j)} \right] - B(0, T_{n+1}) \right\} .
\]

The above equation, applying the change of numeraire technique [8], can be written as

\[
C = \sum_{n=1}^{N} \left\{ B(0, T_n) \mathbb{E}^{(T_n)} \left[ \min_{j=0, \ldots, n} \frac{B_n(T_n)}{B_j(T_j)} \right] - B(0, T_{n+1}) \right\} (9)
\]

where \( \mathbb{E}^{(T_n)} [\cdot] \) is the expectation under the forward measure \( Q^{(T_n)} \) and

\[
W^{(T_n)}_t \equiv W_t - \int_0^T \rho \sigma(t, T_n) dt
\]

is martingale under the forward measure \( Q^{(T_n)} \).

Let us consider the case where the lowest value in equation (9) is the \( j^{th} \) element, in this case the condition reads

\[
\frac{B_n(T_n)}{B_j(T_j)} < \frac{B_n(T_n)}{B_i(T_i)} \quad \forall i \neq j
\]

and (9) becomes

\[
C = \sum_{n=1}^{N} \left\{ B(0, T_n) \sum_{j=0}^{n} \mathbb{E}^{(T_n)} \left[ \frac{B_n(T_n)}{B_j(T_j)} \right] \frac{B_n(T_n)}{B_j(T_j)} - \frac{B_n(T_n)}{B_i(T_i)} \quad \forall i \neq j \right\} .
\]

(10)
After some algebra, it is straightforward to show that

\[
\frac{B_n(T_n)}{B_i(T_i)} = \frac{B_n(0)}{B_i(0)} K^{(n,i)} e^{-\frac{1}{2} \int_0^{T_n} g_i^{(n)}(t) \rho g_i^{(n)}(t) dt + \int_0^{T_n} g_i^{(n)}(t) dW_i^{(T_n)}}
\]

with

\[
g_i^{(n)}(t) \equiv \begin{bmatrix} v_n(t) - v_i(t) & t \in [0, T_i) \\ v_n(t) & t \in [T_i, T_n] \end{bmatrix}
\]

and

\[
\eta_{il}^{(n)} \equiv Corr \left[ \ln \frac{B_n(T_n)}{B_i(T_i)}, \ln \frac{B_n(T_n)}{B_l(T_i)} \right] \quad 0 \leq i \leq l < n.
\]

The theorem is proven applying the Girsanov transform

\[
W_i^{(T_n,j)} = W_i^{(T_n)} - \int_0^{T_n} \rho g_j^{(n)}(t) dt
\]

to the \( j \)th integral of equation (10) \( \forall j < n \)

\[
\mathcal{C} = \sum_{n=1}^{N} \left\{ B(0, T_n) \frac{B_n(0) K^{(n,j)}}{B_j(0)} \left[ B_n(T_n) - B_i(T_n) \right] \right\} + \sum_{j=0}^{n-1} \left\{ B(0, T_n) K^{(j,j)} - \frac{B_n(T_n)}{B_j(T_i)} \left[ 1 - \frac{B_n(T_n)}{B_i(T_i)} \right] \right\}.
\]

Let us briefly recall some interesting features related to the calibration of MHJM discussed in detail in [4]. As shown in [4] the MHJM Model is of particular interest since an analytical solution for caplet and floorlet is available in the plain vanilla case [9, 7] and then it allows to calibrate the model to market volatilities.

Let us consider an \( N \) factor model where only the \( i \)th component of the volatility \( v_i(t) \) is different from zero, i.e. \( (v_i(t))_t = \nu_i(t) \delta_{i,l} \) where \( \nu_i(t) \) is one dimensional function of time and \( \delta_{i,l} \) is Dirichlet delta.

The volatilities \( \nu_i(t) \) can be calibrated directly on market data, using plain vanilla caplets and floorlets. The \( i \)th plain vanilla caplet (with maturity in \( T_i \) and payment in \( T_{i+1} \)) with
strike $K$ is equal to

$$c_i \equiv \theta E[e^{-\int_0^{T_i+1} r_i dt} (L_i(T_i) - K)^+] =$$

$$= B(0, T_{i+1})[(1 + \theta L_i(0)) N(d_1^{(PV)}) - (1 + \theta K) N(d_2^{(PV)})] ,$$

(12)

and the $i^{th}$ floorlet with strike $K$ is

$$f_i \equiv \theta E[e^{-\int_0^{T_i+1} r_i dt} (K - L_i(T_i))^+] =$$

$$= B(0, T_{i+1})[(1 + \theta K) N(-d_2^{(PV)}) - (1 + \theta L_i(0)) N(-d_1^{(PV)})]$$

(13)

where

$$d_1^{(PV)} = \frac{1}{\nu_i \sqrt{T_i}} \ln \frac{1 + \theta L_i(0)}{1 + \theta K} + \frac{1}{2} \nu_i \sqrt{T_i} ,$$

$$d_2^{(PV)} = \frac{1}{\nu_i \sqrt{T_i}} \ln \frac{1 + \theta L_i(0)}{1 + \theta K} - \frac{1}{2} \nu_i \sqrt{T_i}$$

with

$$\nu_i^2 \equiv \frac{1}{T_i} \int_0^{T_i} \nu_i^2(t) dt .$$

(14)

The correlation $\rho_{i,l}$ can be calibrated directly on market data since

$$\rho_{i,l} = Corr[d \ln B_i(t), d \ln B_l(t)] .$$

(15)

In this letter we have derived an analytical solution for both Sticky Cap and Sticky Floor in the Multi Factor Gaussian HJM framework; we have also shown how to calibrate the model with market data (rates, correlations and the term structure of vols).

In the MHJM model it is extremely easy the calibration since we have separate vols $\nu_i(t)$ (equations (12) and (13)) and correlations $\rho_{il}$ (equation (15)): as already stressed by [10], the ability to separately fit volatilities and correlations in Fixed Income derivatives greatly simplifies model calibration to market prices. The analytical approach proposed, besides the advantage of being easy to calibrate, offers closed form solutions in pricing Fixed Income contingent claims even when path dependence features are involved.
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References


