Smile and multi-factor modelling for exotic products

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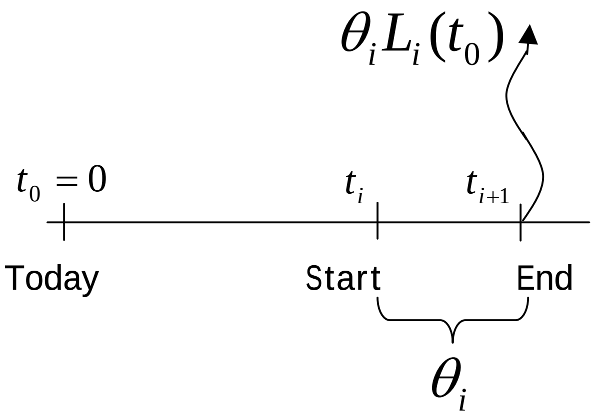
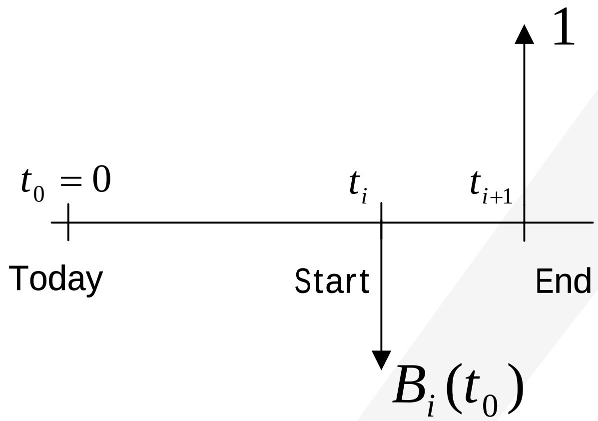
Outline

- Multifactor modelling…
  - ✓ Bond Market Model vs Libor Market Model
  - ✓ Cap/Floor & Swaption Formulas
  - ✓ Monte Carlo simulations & Markov Property
- … when smile is present
  - ✓ Smile & Sticky Delta
  - ✓ Brigo-Mercurio-Rapisarda vs Gigi model
  - ✓ Calibration & MonteCarlo
- Bermudan options
  - ✓ Formulation
  - ✓ Lower Bound: Standard Approach
  - ✓ Lower Bound: Perturbation Approach
  - ✓ Upper Bound
  - ✓ Examples

Forward Libor Rates (in t_0) $L_i(t_0)$



Forward ZC Bond (in t_0) $B_i(t_0)$



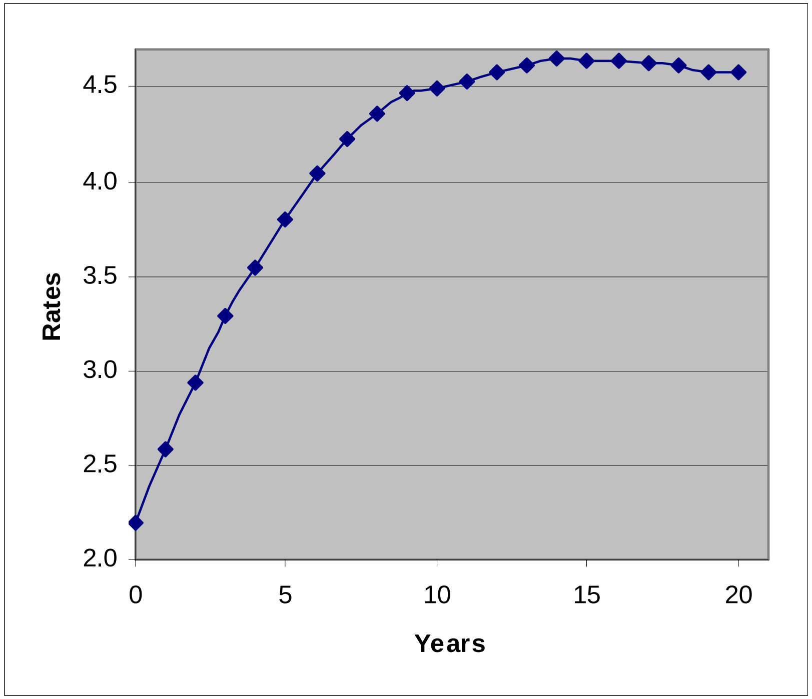
... and their relation

$$L_i(t_0) = \frac{1}{\theta_i}\left\{\frac{1}{B_i(t_0)} - 1\right\}$$

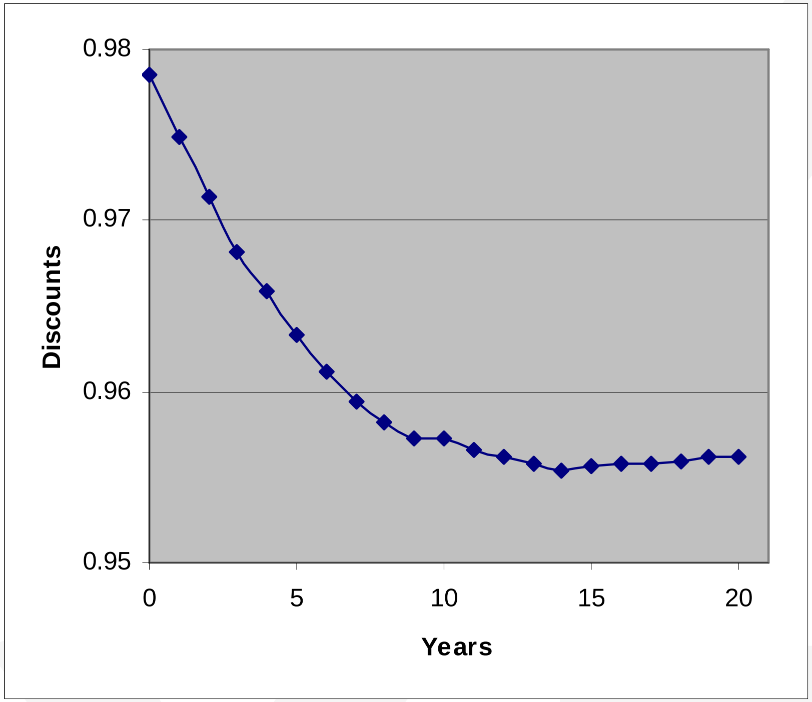
...

IR Markets: Underlying

Rates $L_i(T_0)$



ZC Bonds $B_i(T_0)$



A possible model…

$$dL_i(t) = (\cdots)dt - L_i(t)\sigma_i dW(t)$$

… and another possibility

$$dB_i(t) = (\cdots)dt + B_i(t)v_i dW(t)$$

Dataset: 14 Jan 05 at 11:15 CET

LMM Dynamics: spot measure

$$dL_i(t) = L_i(t)\sigma_i \left\{ \sum_{j=k+1}^{i} \rho_{ij}^{(L)} \sigma_j \frac{\theta_j L_j(t)}{1+\theta_j L_j(t)} dt - dW(t) \right\}$$

with $t_k \le t < t_{k+1}$; $dW_i(t) \bullet dW_j(t) = \rho_{ij}^{(L)} dt$

$\sigma_i = 0 \quad for \quad t \ge t_i$ ⇨ Fixing Mechanism

LMM Dynamics: t_{i+1} forward measure

$$dL_i(t) = -L_i(t)\sigma_i dW^{(i+1)}(t)$$

BMM Dynamics: spot measure

$$dB_i(t) = B_i(t)v_i\left\{-\sum_{j=k+1}^{i}\rho_{ij}^{(B)}v_j dt + dW(t)\right\}$$

$$\text{with}\quad t_k \le t < t_{k+1} \quad ; \quad dW_i(t) \bullet dW_j(t) = \rho_{ij}^{(B)} dt$$

$$v_i = 0 \quad for \quad t \ge t_i$$ ⇨ Fixing Mechanism

BMM Dynamics: t_i forward measure

$$dB_i(t) = B_i(t)v_i dW^{(i)}(t)$$

Models: Caplet (Exact) Solutions

Libor Model

$$c_i = B^0_{0i+1}\theta_i\left\{L_i(t_0)\bullet N(d^{cf}_+) - K\bullet N(d^{cf}_-)\right\}$$

$$d^{cf}_+ = \frac{1}{\sigma_i\sqrt{t_i}}\ln\left(\frac{L_i(t_0)}{K}\right) + \frac{1}{2}\sigma_i\sqrt{t_i} \quad ; \quad d^{cf}_- = d^{cf}_+ - \sigma_i\sqrt{t_i}$$

Bond Model

$$c_i = B^0_{0i+1}\left\{\left[1+\theta_i L_i(t_0)\right]\bullet N(d^{(B)}_+) - \left[1+\theta_i K\right]\bullet N(d^{(B)}_-)\right\}$$

$$d^{(B)}_+ = \frac{1}{v_i\sqrt{t_i}}\ln\left(\frac{1+\theta_i L_i(t_0)}{1+\theta_i K}\right) + \frac{1}{2}v_i\sqrt{t_i} \quad ; \quad d^{(B)}_- = d^{(B)}_+ - v_i\sqrt{t_i}$$

Libor Model

- Exact: *Not Available*
- Approximated: $SP_{\alpha\omega} = B_{0\alpha}(T_0) BPV_{\alpha\omega}(T_0) \left\{ S_{\alpha\omega}(T_0) \bullet N(d_1^s) - k \bullet N(d_2^s) \right\}$

$$d_1^s = \frac{1}{\sigma_{\alpha\omega}\sqrt{T_\alpha}} \ln\left(\frac{S_{\alpha\omega}(T_0)}{k}\right) + \frac{1}{2}\sigma_{\alpha\omega}\sqrt{T_\alpha}; \qquad d_2^s = d_1^s - \sigma_{\alpha\omega}\sqrt{T_\alpha};$$

$$\sigma_{\alpha\omega}^2 = \sum_{i,j=\alpha}^{\omega-1} \eta_i \sigma_i \rho_{ij}^{(L)} \eta_j \sigma_j$$

Not so good…

Bond Model

- Exact: *Available* (see e.g. Musiela Rutkowski 1997)
- Approximated: $SP_{\alpha\omega} = B_{0\alpha}(T_0) \left\{ N(-d_2^B) - P_{\alpha\omega}(k;T_0) \bullet N(-d_1^B) \right\}$

$$d_1^B = \frac{1}{V_{\alpha\omega}\sqrt{T_\alpha}} \ln\left(P_{\alpha\omega}(k;T_0)\right) + \frac{1}{2} V_{\alpha\omega}\sqrt{T_\alpha}; \qquad d_2^B = d_1^B - V_{\alpha\omega}\sqrt{T_\alpha}$$

$$V_{\alpha\omega}^2 = \sum_{i=\alpha+1}^{\omega} \gamma_i v_i \rho_{ij}^{(B)} \gamma_i v_i \quad \& \quad P_{\alpha\omega}(k;T_0) = k \sum_{i=\alpha+1}^{\omega} B_{\alpha i}(T_0) + B_{\alpha\omega}(T_0)$$

Precision… let we see

Approximation Precision BMM solution

Our goal is to measure only the error
coming from the *approximation* of swaption formula

We plot the difference between the exact and the approximated solution
in the Euro market with a correlation

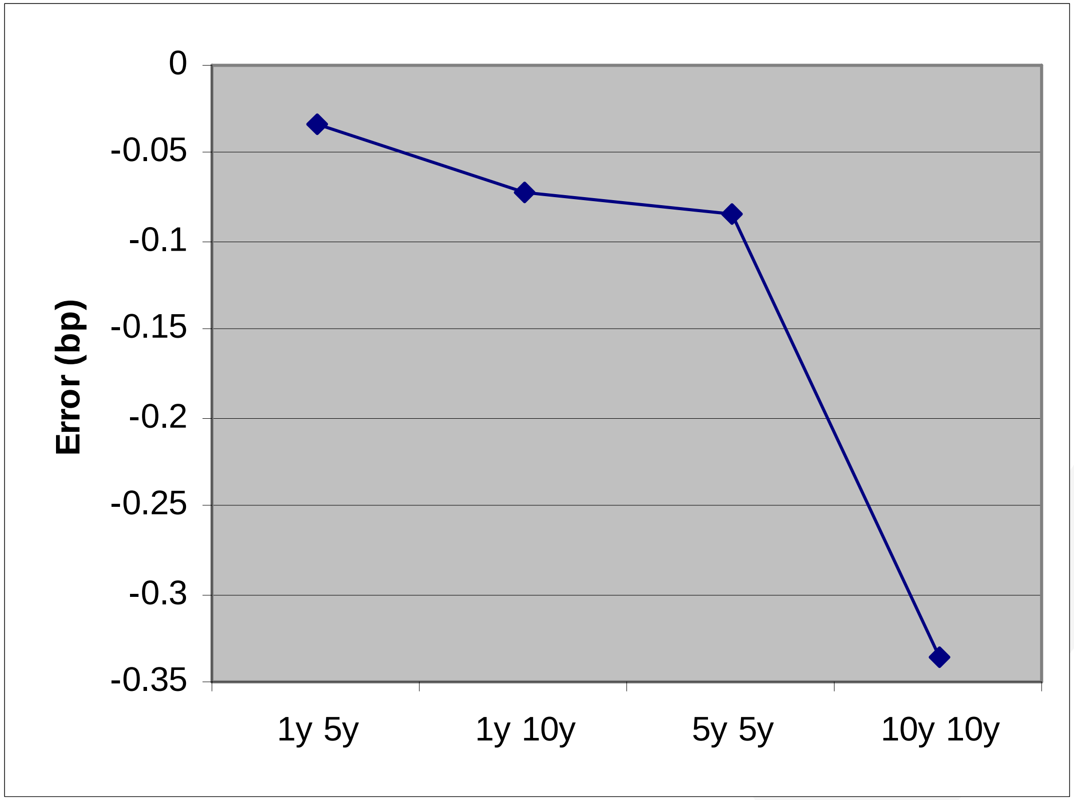
$$\rho_{ij}^{(B)} = Exp(-a\,|\,i - j\,|); \qquad a = 8 \bullet 10^{-2}$$

and the cap/floor volatilities of the 14th January 2005 at 11:15 CET.

We consider some cases:

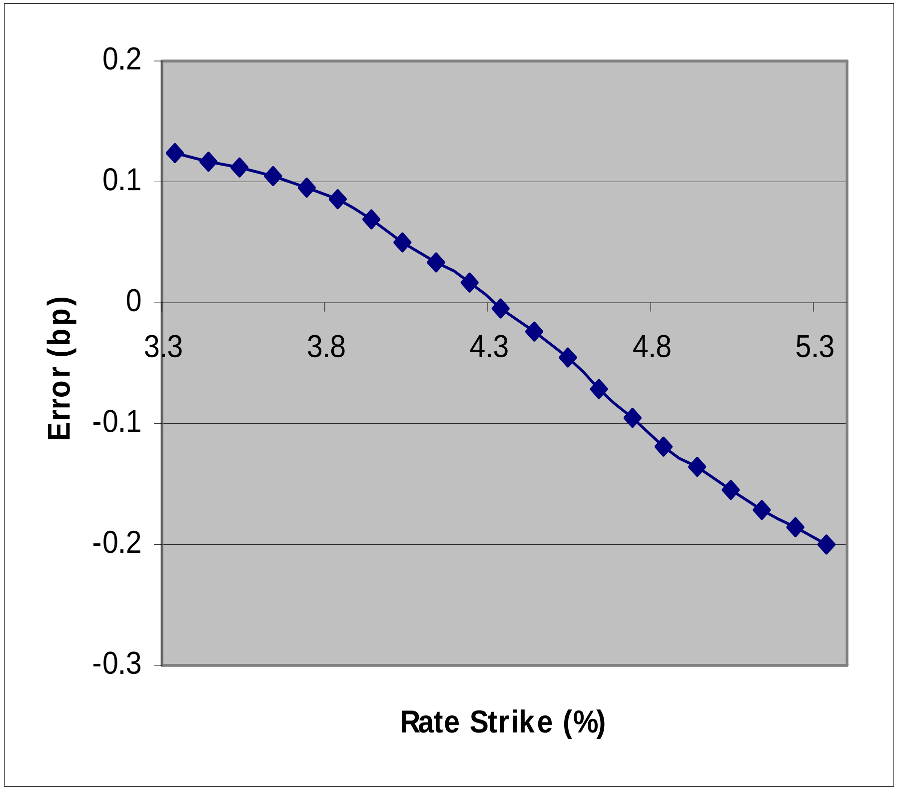
- ATM straddle
- Some very liquid swaptions *vs* Strike (5y 5y & 1y 5y)

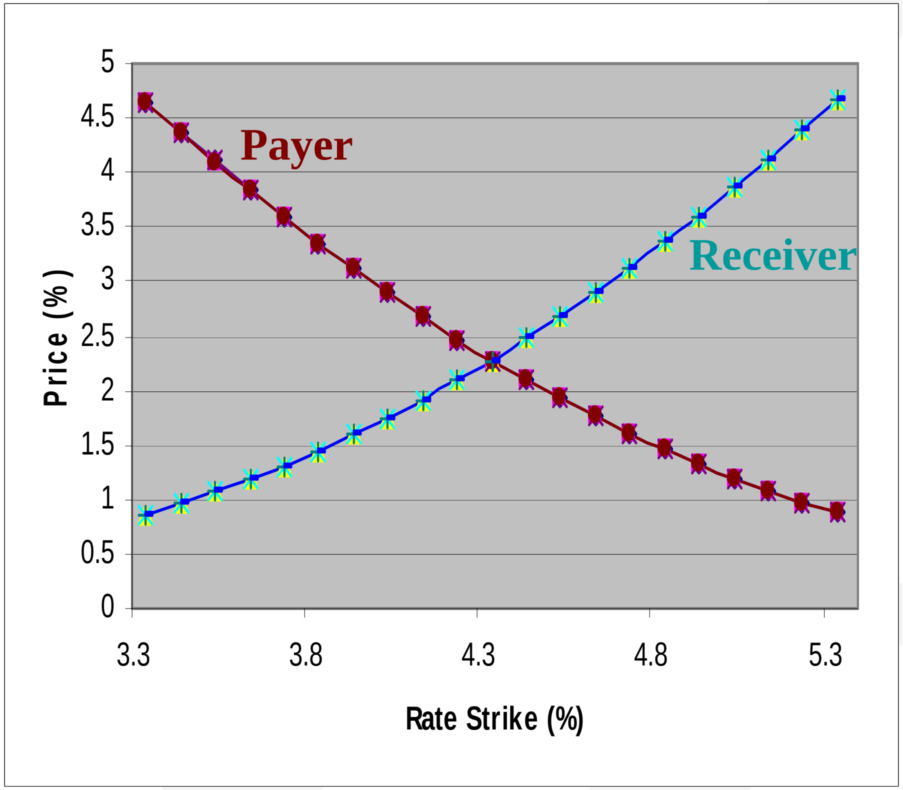
Approximation Precision: ATM Straddle



1 bp = 0.01 %

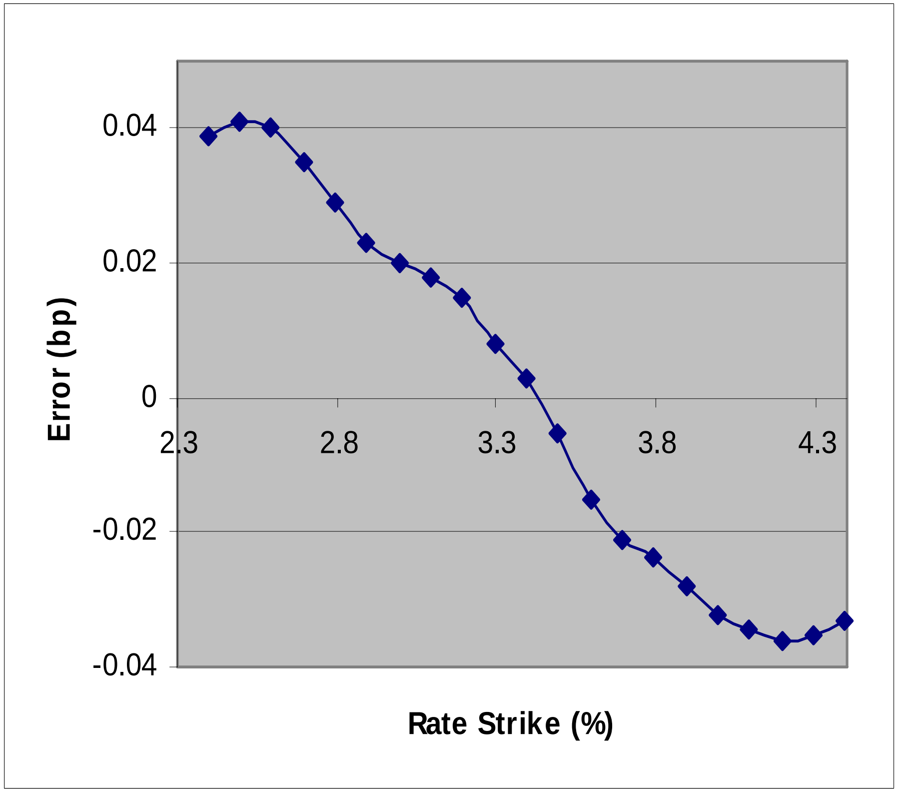
Approximation Precision: 5y 5y

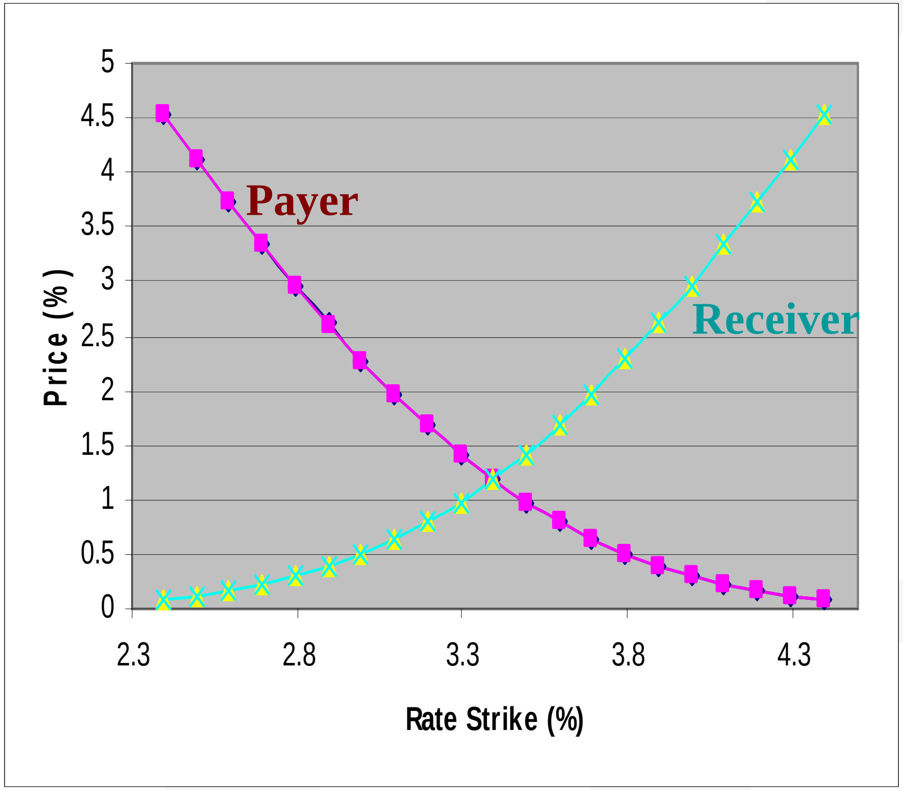




1 bp = 0.01 %

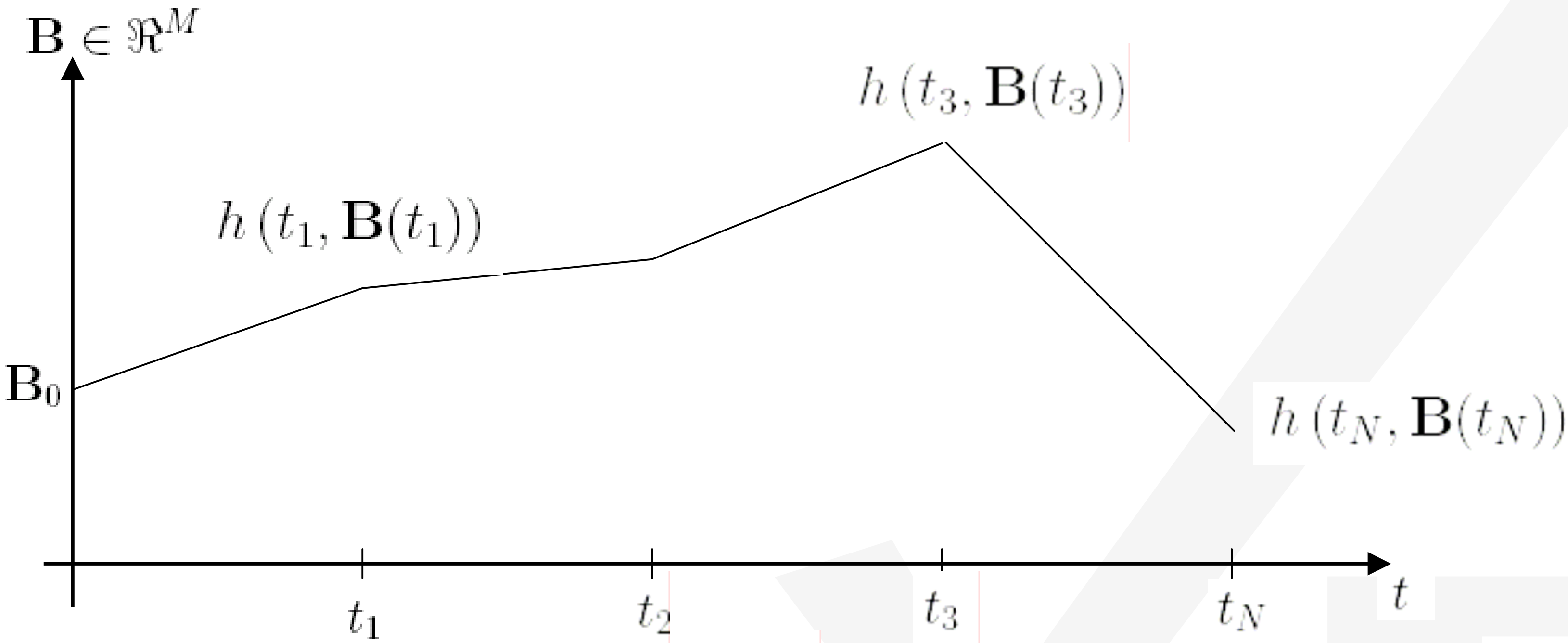
Approximation Precision: 1y 5y





1 bp = 0.01 %

MonteCarlo: std approach for Non-Callable products



$$O = E\left[\sum_i D_{0\,i} h(t_i, \mathbf{B}(t_i))\right]$$

$D_{0\,i}$: discount in (t_0, t_i)

$h(t_i, \mathbf{B}(t_i))$: payoff in t_i

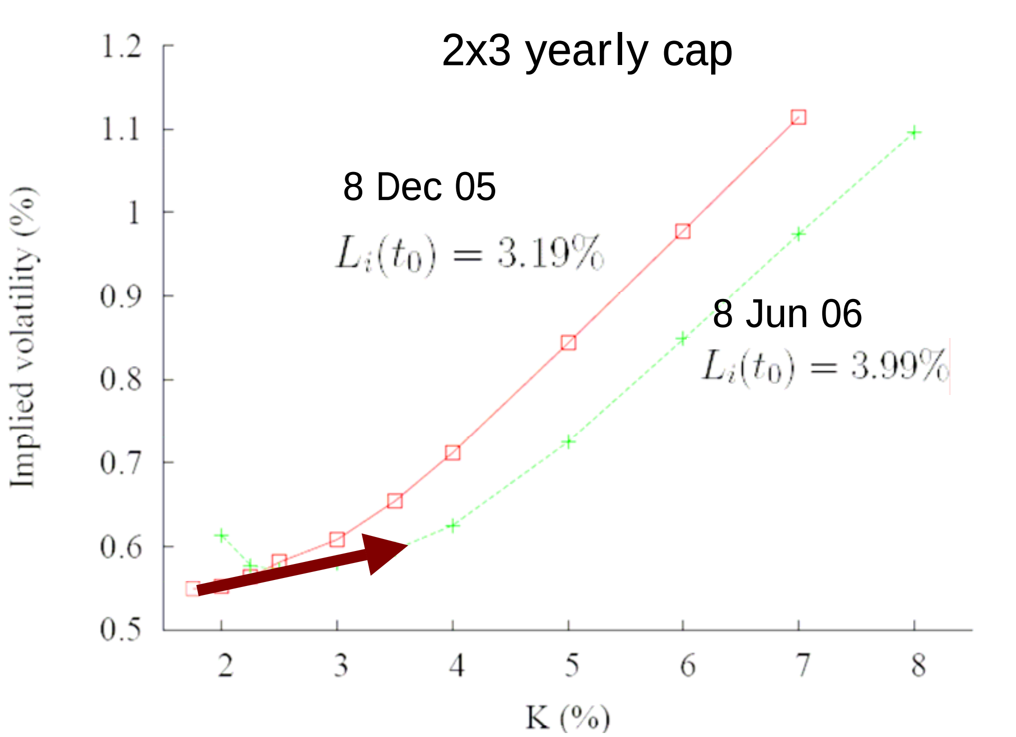
Why MonteCarlo? Lattice methods work poorly for high-dimentional problems.

Main Properties:

- Elementary Monte Carlo: $\forall$ fwd measure! (Gaussian HJM)
- Markov Chain between Reset dates $\{\mathbf{B}(t_i)\}_i$: (Lognomal) Transition Probability
- $D_{0\ i}$ function of $\{\mathbf{B}(t_i)\}_i$: $D_{0\ i} = \prod_{n=0}^{i-1} B_n(t_n)$
- Cap/Floor & Swaption: Black-like formulas
- A large set of analytical formulas Exact or Almost Exact
- …

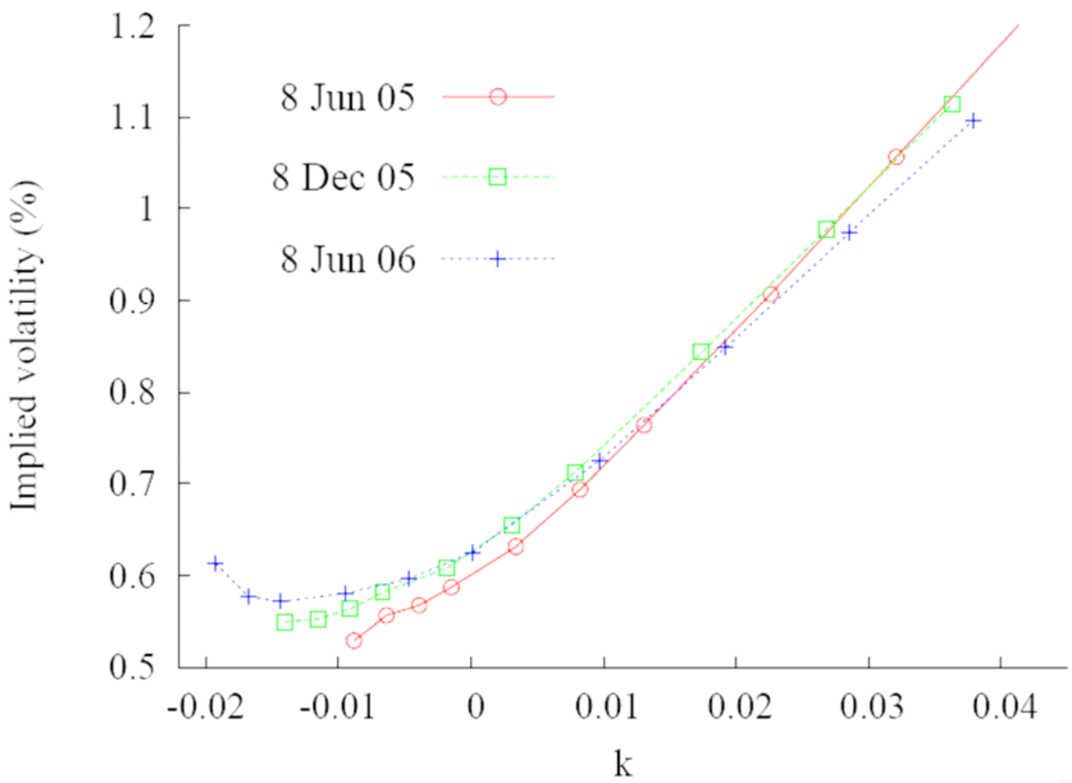
Smile & Sticky Delta

Implied price-vol & ATM Libor Rates



Move in the same direction

Sticky Delta: how to model



Prices depend from the moneyness

$$k \equiv \ln \frac{1 + \theta_i K}{1 + \theta_i L_i(t_0)}$$

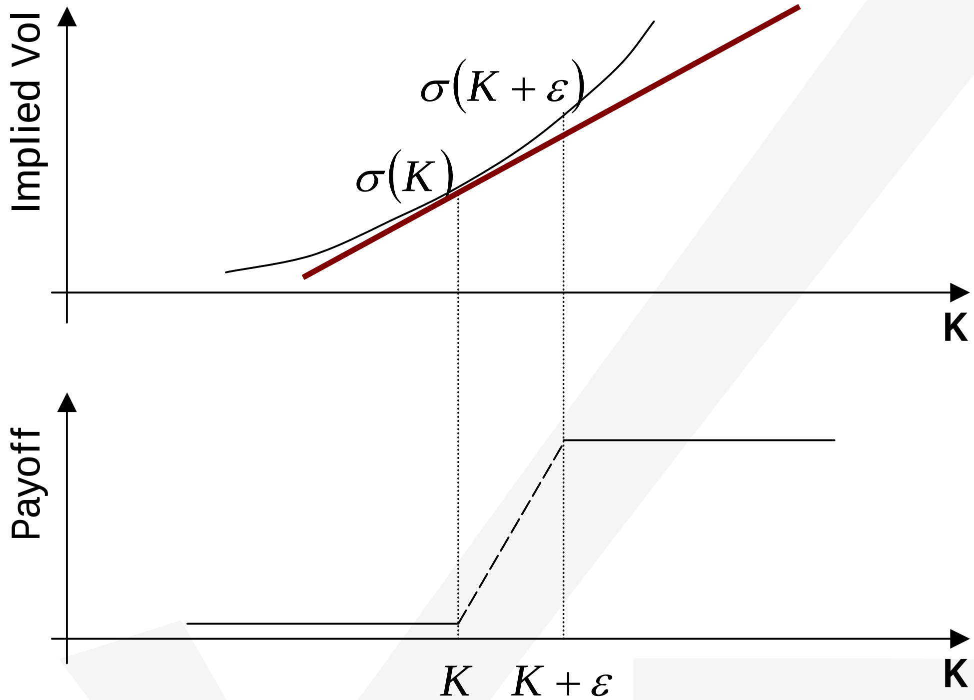
Why so relevant when modelling rates: Digital options

Implied volatility approach

$$dc = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ c\left[K, \sigma(K)\right] - c\left[K+\varepsilon, \sigma(K+\varepsilon)\right] \right\} =$$

$$-\frac{d}{dK} c\left[K, \sigma(K)\right] =$$

$$-\frac{\partial}{\partial K} c\left[K, \sigma(K)\right] - \frac{\partial \sigma(K)}{\partial K} \frac{\partial c\left[K, \sigma\right]}{\partial \sigma}$$



Slope Impact ⇨ Digital "*detects*" the difference *Black* vs *Implied*

see Gatheral (2006)

Basic idea: Brigo Mercurio Rapisarda

In the t_i fwd measure…

$$\ln \frac{B_i(t_i)}{B_i^0} = -\frac{1}{2} t_i \, v_i^2 + \sqrt{t_i} v_i g_i$$

g_i : standard normal

BMM: v_i model parameter ⇒ $c_i^B(k; v_i) = B_{0i}^0 \, \left[N\left[d_1(k; v_i)\right] - e^k N\left[d_2(k; v_i)\right]\right]$

BMR: v_i r. v. ⇒ $c_i^G(k) = \mathbb{E}_v\left[c_i^B(k; v_i)\right]$

with $k \equiv \ln \frac{1+\theta_i K}{1+\theta_i L_i(t_0)}$ $\quad d_{1,2}(k; v_i) = -\frac{k}{\sqrt{(t_i - t_0) v_i^2}} \pm \frac{1}{2}\sqrt{(t_i - t_0) v_i^2}$

e.g. v_i — hi vol / mid vol / low vol

… as in scenario analysis

see Brigo et Al. (2004)

Gigi model

In the t_i fwd measure…

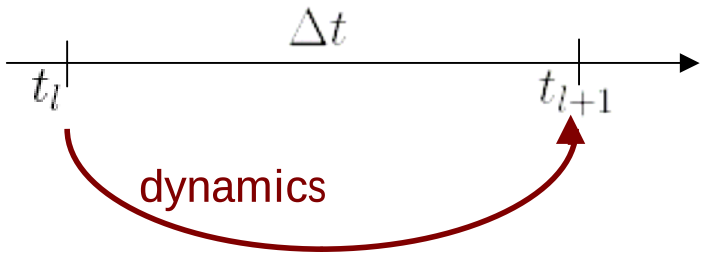
$$\ln \frac{B_i(t_i)}{B_i^0} = -\left(\frac{1}{2} + \eta_i\right)\ t_i\ v_i^2 + \sqrt{t_i} v_i g_i - t_i \psi_i\,[\eta_i]$$

$G_i\ g_i$

$$\begin{cases} v_i^2 = \nu_i^2 G_i \\ G_i : \quad IG\left(1, \frac{\kappa_i}{t_i}\right) \\ \psi_i[\omega] = \frac{1}{\kappa_i}\left[1 - \sqrt{1 + 2\nu_i^2 \kappa_i \omega}\right] \end{cases}$$

g_i : standard normal

$$Corr(g_i, g_n) = \rho_{in}, \quad \overline{g_i\ G_n} = 0, \quad \overline{G_i\ G_n} = 0$$



$$(l < i)$$

In the t_i fwd measure…

$$\ln \frac{B_i(l+1)}{B_i(l)} = -\Delta t \left\{ \left(\frac{1}{2} + \eta_i \right) v_i^2(l) + \psi_i \left[\frac{\eta_i}{2} \right] \right\} + \sqrt{\Delta t}\, v_i(l) g_i(l)$$

with

$$\begin{cases} v_i^2(l) = \nu_i^2 G_i(l) \quad (l < i) \\ G_i(l) : IG\left(1, \frac{\kappa_i}{\Delta t}\right) \\ \psi_i[\omega] = \frac{1}{\kappa_i}\left[1 - \sqrt{1 + 2\nu_i^2 \kappa_i \omega}\right] \end{cases}$$

Markov chain dynamics

$\forall\, t_j$-fwd measure $j \leq i$

Gigi Calibration: closed formulas

Caplets

$$c_i^G(k) = \mathbb{E}_v\left[c_i^B(k(v_i); v_i)\right]$$

with $k(v) \equiv k - \eta_i t_i \, v^2 - t_i \psi_i[\eta_i]$

… and also

$$c_i^G(k) = B_{0i}^0 \left\{ 1 - \int_{i\omega-\infty}^{i\omega+\infty} \frac{dz}{2\pi} e^{-izk} \frac{\phi_i(-z)}{z^2 - iz} \right\}$$

Simple 1-d integrals

with $\omega \in (0,1)$ and $\phi_i(z)$ return's characteristic function

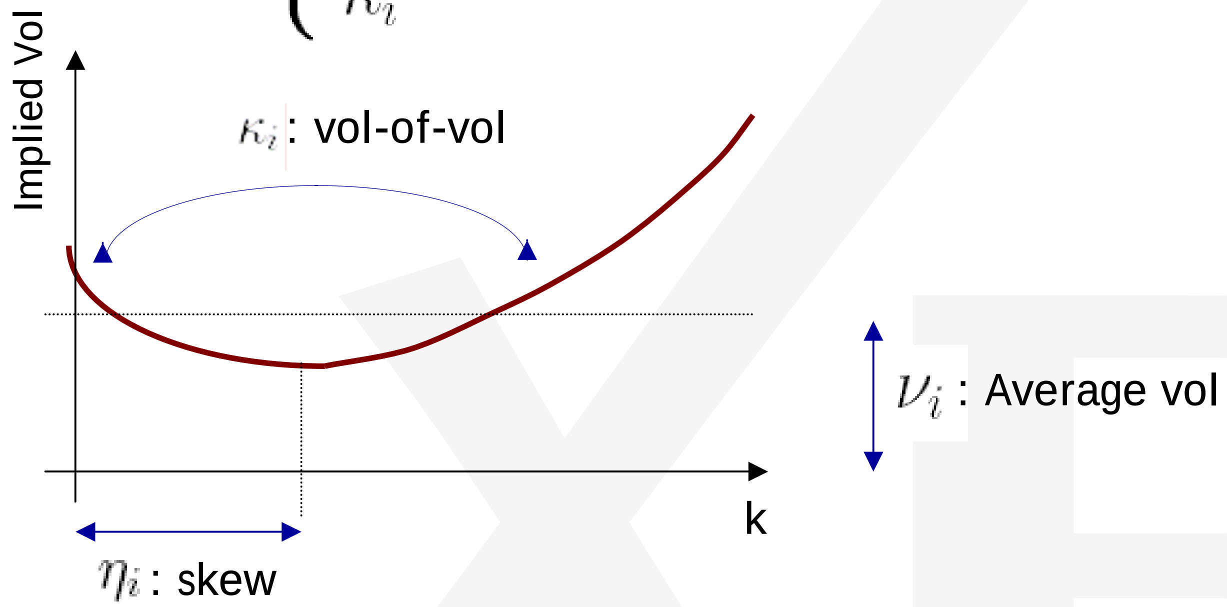
$$\phi_i(z) = \exp\left\{-iz\, t_i\, \psi_i[\eta_i]\right\} exp\left\{ t_i\, \psi_i \left[\frac{z^2 + i(1+2\eta_i)z}{2} \right] \right\}$$

see Lewis (2004)

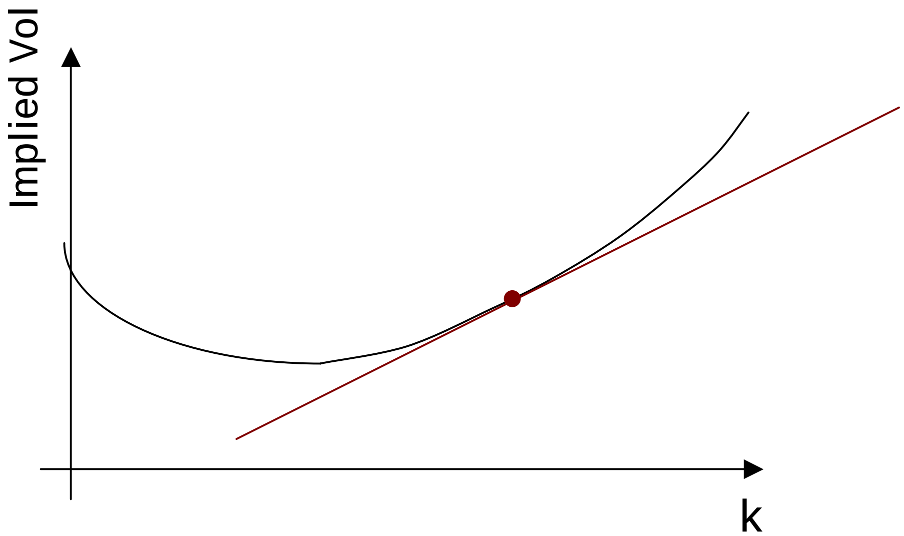
Swaptions: similar results hold…

BMM 1 parameter: implied volatility

Gigi 3 parameters $\forall$ implied curve $\begin{cases} \nu_i \\ \eta_i \\ \kappa_i \end{cases}$

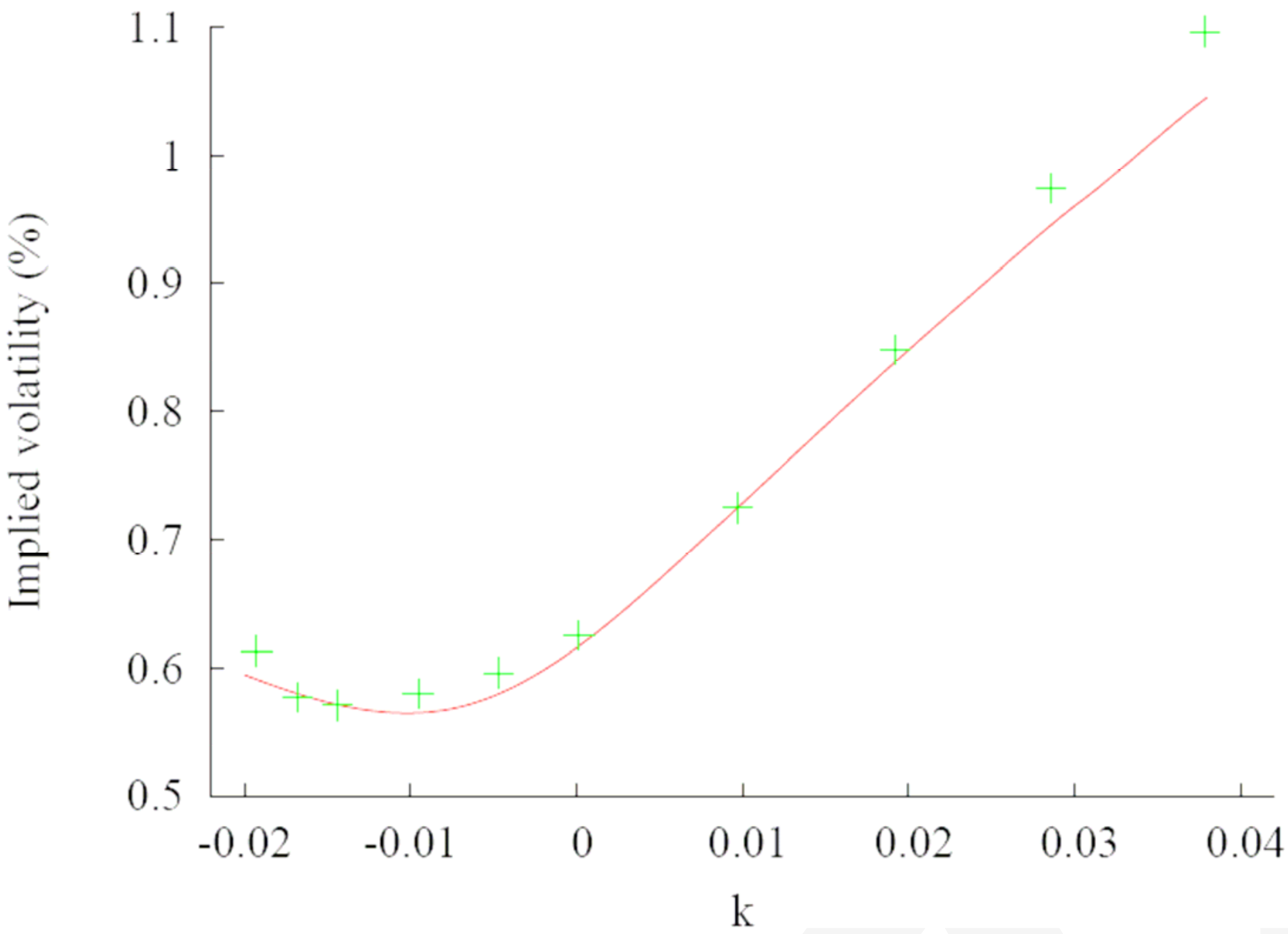


Local calibration

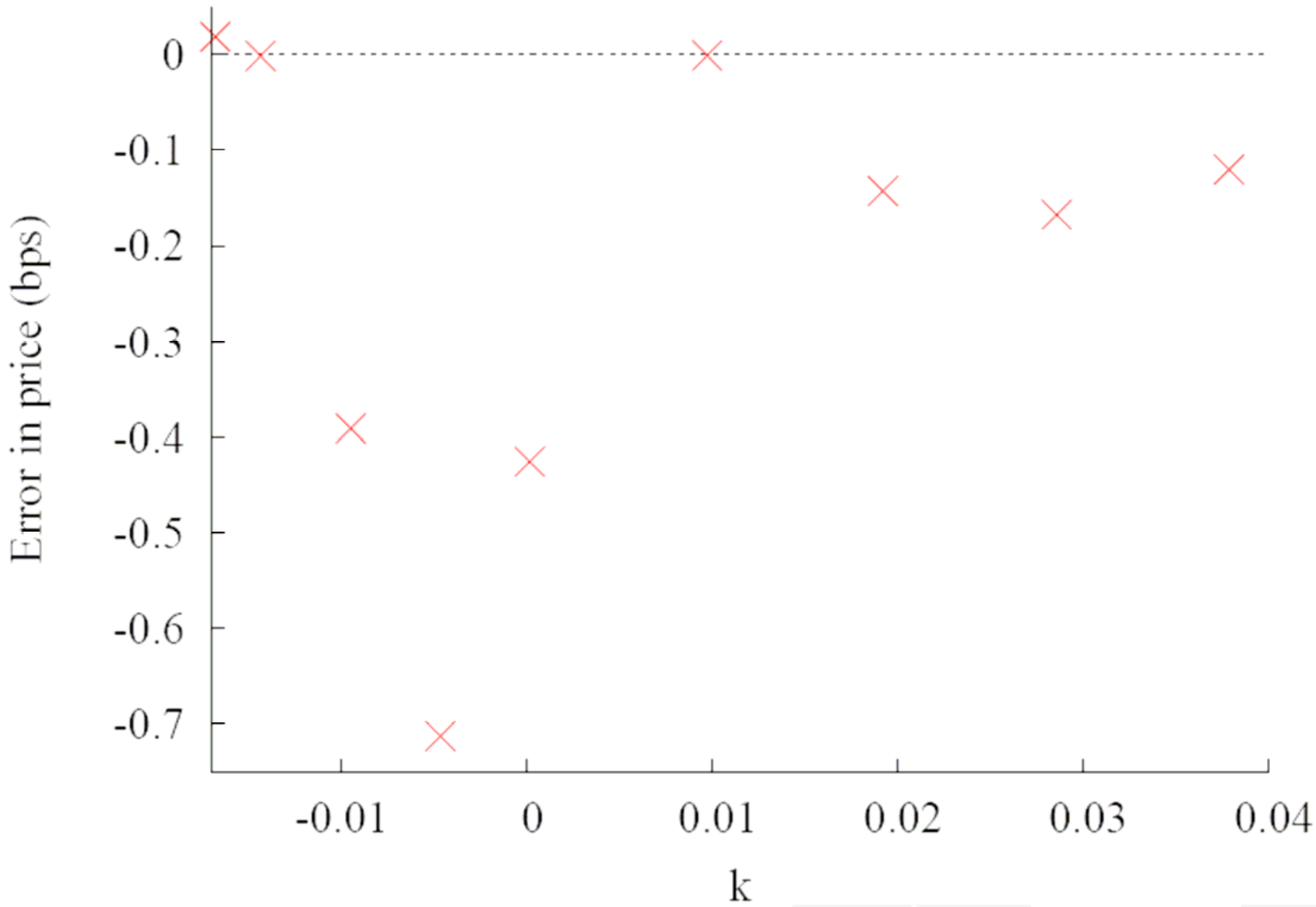


Impose that mkt and calibrated Implied vol have the same slope on the relevant strike ⇨ Reflect digital risk

Numerical results: 8 Jun 06



Numerical results: errors



Summing Up: Gigi an elementary multifactor IR model with smile

- Monte Carlo simulation
  - ✓ Markov chain property between reset dates
  - ✓ Simple transition probability (& easy to implement numerically)

- Calibration:
  - ✓ PV options have 1-d closed form
  - ✓ Reproduces mkt prices for PV (& digitals)
  - ✓ model parameters are easy to be calibrated

- Parsimony
  - ✓ Few parameters describe smile characteristics
  - ✓ they have a financial interpretation

Bermudan option:

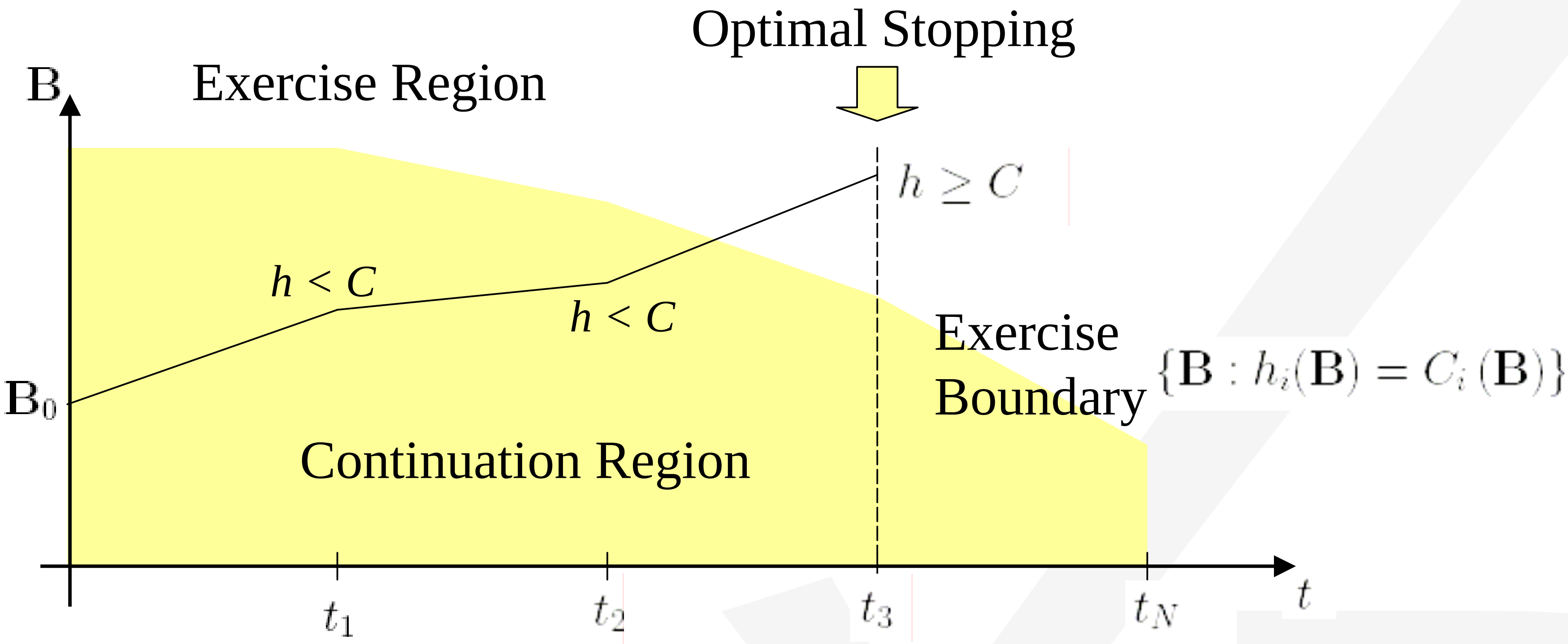
$$C_0 = \sup_{\tau \in \mathcal{T}} E_0 \left[D_{0\tau} \, h\left(\tau, \mathbf{B}(\tau)\right) \right]$$

$\mathcal{T}$: class of admissible stopping times with values in $\{t_1, t_2, \ldots, t_N\}$

Optimal stopping τ^*

$$\tau^* = \min_i \left\{ t_i : h\left(t_i, \mathbf{B}(t_i)\right) \geq C_i\left(\mathbf{B}(t_i)\right) \right\}$$

with $C_i(\mathbf{B})$: Continuation value function



Problem:

$C_i(\mathbf{B})$ is a Bermudan option with exercise dates $\{t_{i+1}, t_{i+2}, \ldots, t_N\}$

In a MC approach each $C_i(\mathbf{B})$ should come from a **new** MC simulation starting in t_i !?!

Any approximate exercise strategy $\hat{\tau}$ provides a lower bound

$$L_0 = E_0\left[D_{0\hat{\tau}}\, h\left(\hat{\tau}, \mathbf{B}(\hat{\tau})\right)\right] \le E_0\left[D_{0\tau^*}\, h\left(\tau^*, \mathbf{B}(\tau^*)\right)\right] = C_0$$

using in the exercise decision an approximation $\hat{C}_i\left(\mathbf{B}, \{\eta\}\right)$

where $\{\eta\}$ are a set of parameters…

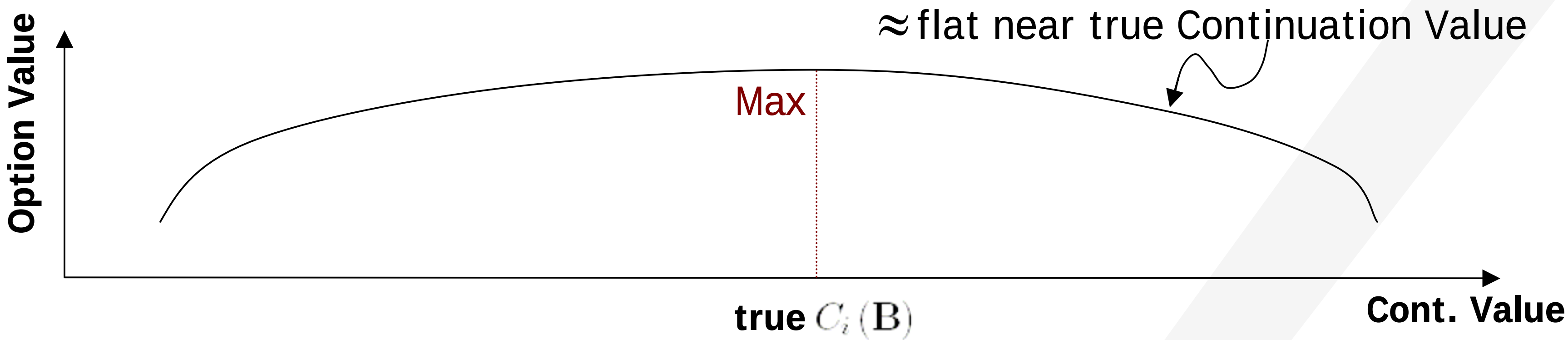
Idea:

Option value *not very sensitive* to the **exact** position of the Exercise Boundary

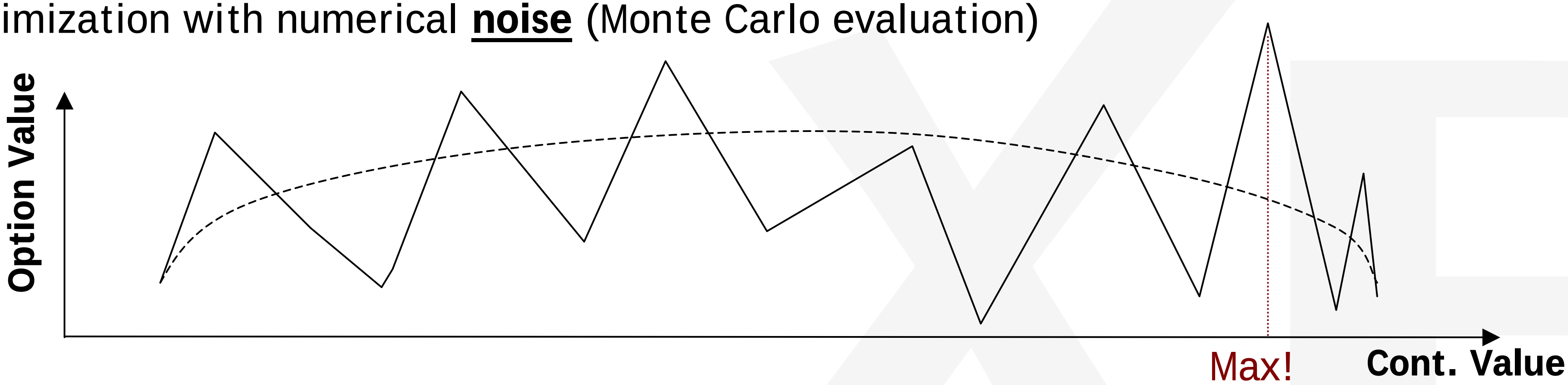
Even a rough approximation of $C_i\left(\mathbf{B}\right)$ leads to a reasonable approximation of option value

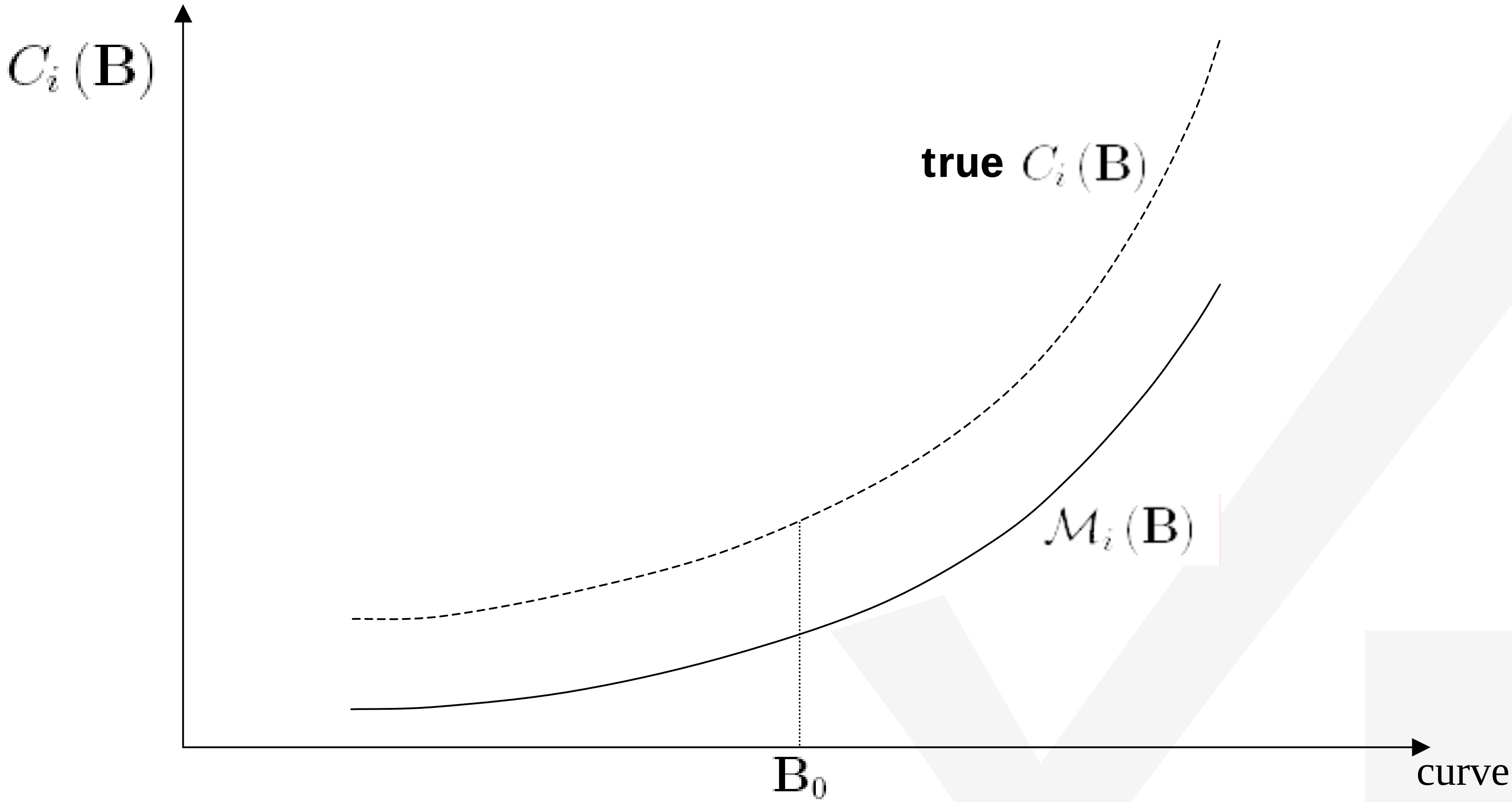
…then find the best $\{\eta\}$. (Andersen 2000)

Optimization **exact** function

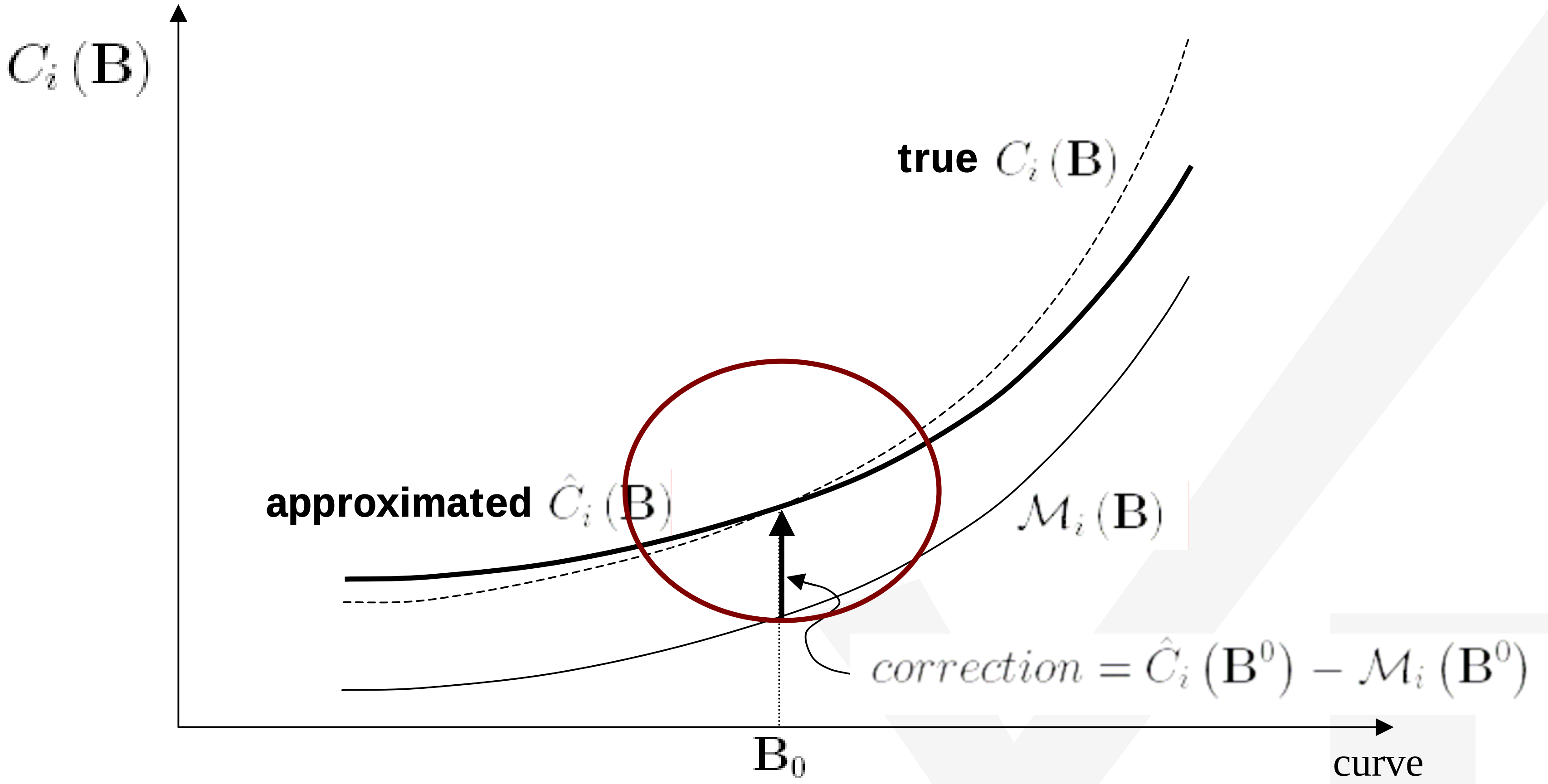


Optimization with numerical **noise** (Monte Carlo evaluation)





with $\mathcal{M}_i(\mathbf{B})$ an arbitrary **simple** (to compute) function



$$\hat{C}_i(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + correction$$

Starting from the (N-1) Continuation value function, already a simple function,

how to get $\hat{C}_i(\mathbf{B})$ knowing $\hat{C}_n(\mathbf{B}) \quad \forall n > i$

$$\hat{C}_n(\mathbf{B}) \quad \forall n > i$$

$$\Downarrow$$

$$\hat{C}_i(\mathbf{B}^0)$$

$$\Downarrow$$

$$\hat{C}_i(\mathbf{B}) \equiv \mathcal{M}_i(\mathbf{B}) + \left(\hat{C}_i(\mathbf{B}_0) - \mathcal{M}_i(\mathbf{B}_0)\right)$$

$\mathcal{M}_i(\mathbf{B})$ a possible choice

$$\mathcal{M}_i(\mathbf{B}) \equiv \mathcal{E}_m(\mathbf{B}, t_i) \qquad 0 < i < m \leq N$$

with $\mathcal{E}_m$ the max European option in $\mathbf{B}^0$: $\mathcal{E}_m(\mathbf{B}^0, t_i) \geq \mathcal{E}_n(\mathbf{B}^0, t_i) \ \forall n$

where $\mathcal{E}_n(\mathbf{B}^0, t_i)$ European option valued in t_i with expiry t_n

$$\hat{C}_i^{\{1\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{1\}}(i)$$

$$\hat{C}_i^{\{2\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{2\}}(i) + \sum_{n=i}^{N} c_1^{\{2\}}(i,n)\left(\ln B_n - \ln B_n^0\right)$$

...

$$c_0^{\{1\}}(i) = \left[\hat{C}_i^{\{1\}} - \mathcal{M}_i\right](\mathbf{B}^0)$$

⇦ value in $\mathbf{B}^0$

$$c_1^{\{2\}}(i,n) = B_n^0 \frac{\partial}{\partial B_n}\left[\hat{C}_i^{\{2\}} - \mathcal{M}_i\right](\mathbf{B}^0)$$

⇦ Delta in $\mathbf{B}^0$

...

$$c_2^{\{3\}}(i,j,n) = \cdots$$

⇦ Gamma in $\mathbf{B}^0$

Idea:

Given Π a class of martingale processes with values in $\{t_1, t_2, \ldots, t_N\}$

$$\sup_{\tau\in\mathcal{T}} E_0 \left[D_{0\tau}\, h\left(\tau, \mathbf{B}(\tau)\right)\right] = C_0 = \inf_{\pi\in\Pi} \left\{ \pi_0 + E_0 \left[\max_i \left(D_{0i} h_i - \pi_i \right) \right] \right\}$$

Lower Bound: L_0

Upper Bound: U_0

(Andersen Broadie 2004 and ref. therein)

An approximated continuation value function set $\left\{\hat{C}_i(\mathbf{B})\right\}_i$

⇩

martingale process $\{\hat{\pi}_i\}_i$

$$\begin{cases} \hat{\pi}_0 = L_0 \\ \hat{\pi}_i = \hat{\pi}_{i-1} + \Delta\pi_i \quad i = 1, .., N \end{cases}$$

with:

$$\Delta\pi_i = D_{0i+1}\left\{\max\left(h_{i+1}, \hat{C}_{i+1}\right) - E_i\left[\max\left(h_{i+1}, \hat{C}_{i+1}\right)\right]\right\} \quad i = 1, ..., N-1$$

$$\Delta\pi_N = D_{0N}h_N - D_{0N-1}\hat{C}_{N-1} \quad i = N$$

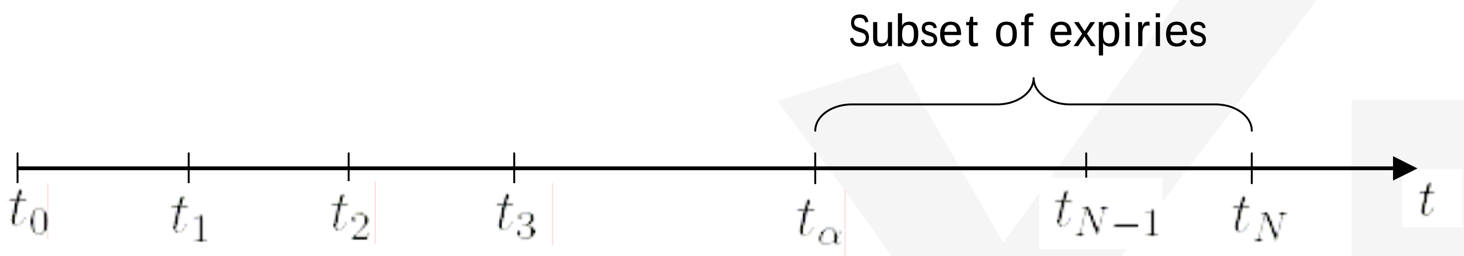
...two nested MCs

Examples

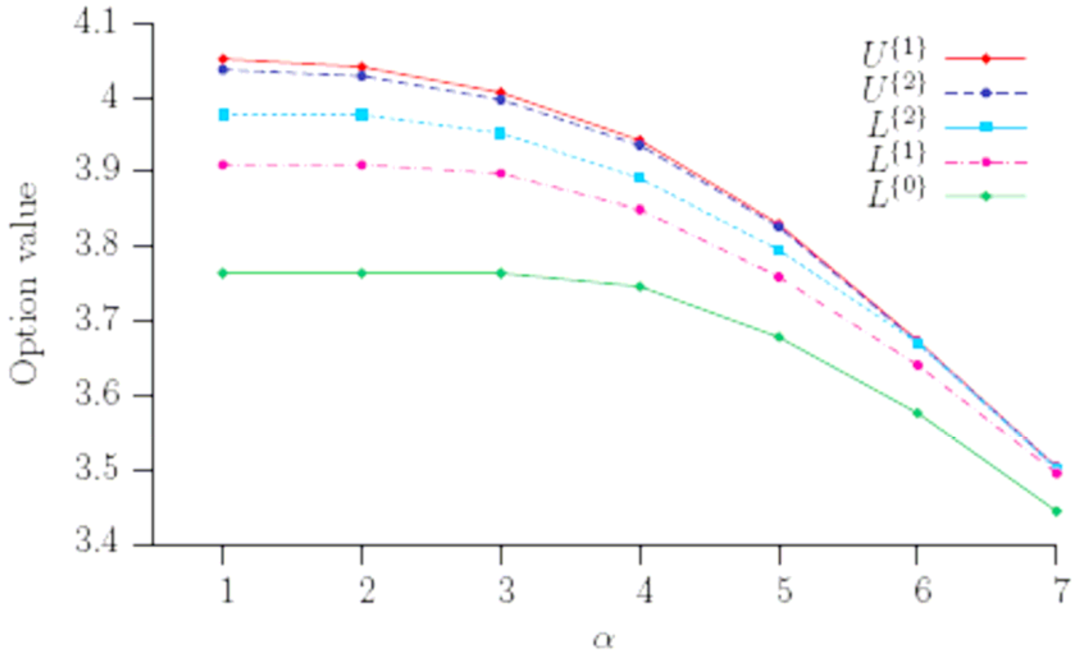
1. 10y S/A ZC Bermudan option (N = 19)
2. 10y A/A Bermudan option on a 10-2 CMS spread (N = 9)

We also consider Lower and Upper bounds for Bermudans with a subset of exercise dates

L_α , U_α : first expiry in t_α



Example 1: ZC Bermudan Option



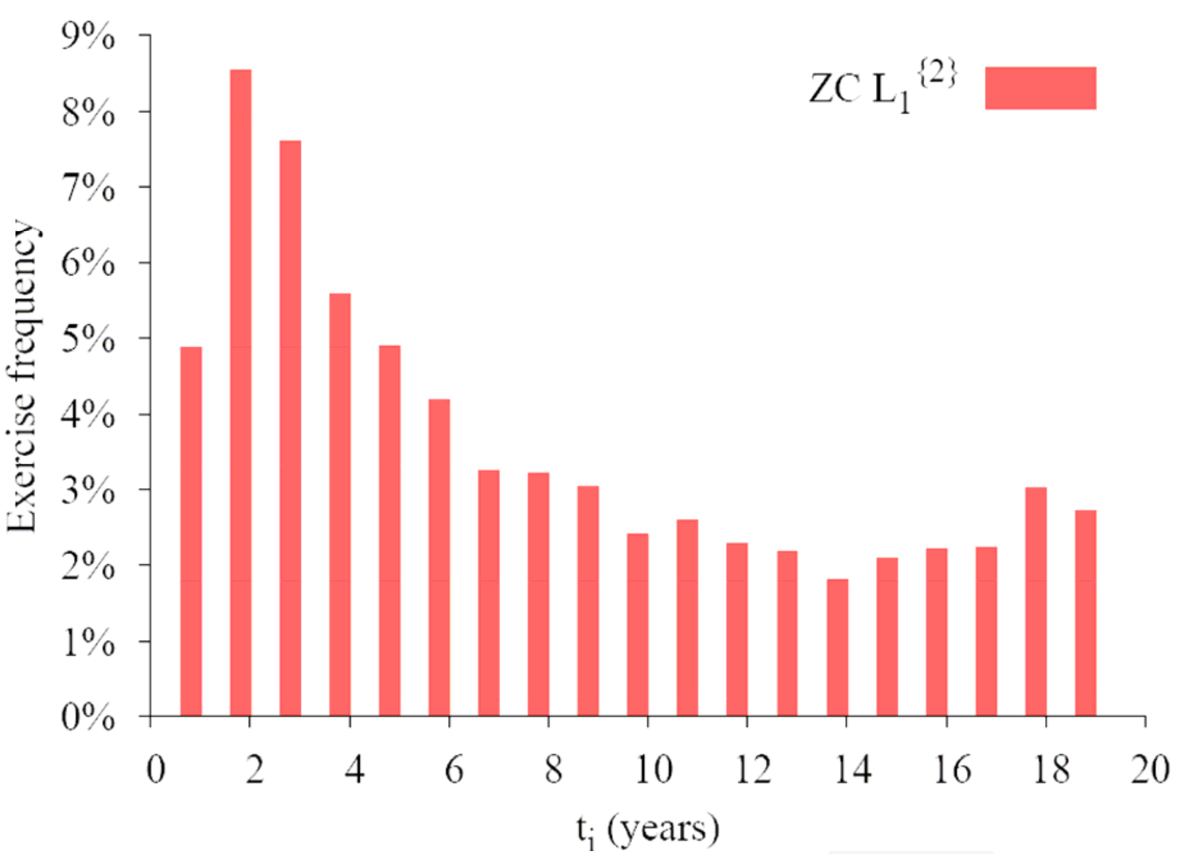
L_α using 10^6 paths

U_α using $5 \cdot 10^4$ paths (external MC) & 10^3 paths (internal MC)

Strikes (N=19): $K_i = 8.75 \cdot 10^{-4}\, i^2 + 1.09 \cdot 10^{-2}\, i + 0.432$

Dataset: 14 Jan 05 at 11:15 CET

Example1: Exercise Frequency



New Approach: Accuracy

Option value $$C_\alpha = L_\alpha + \frac{A}{2}$$

Accuracy in bps(*)

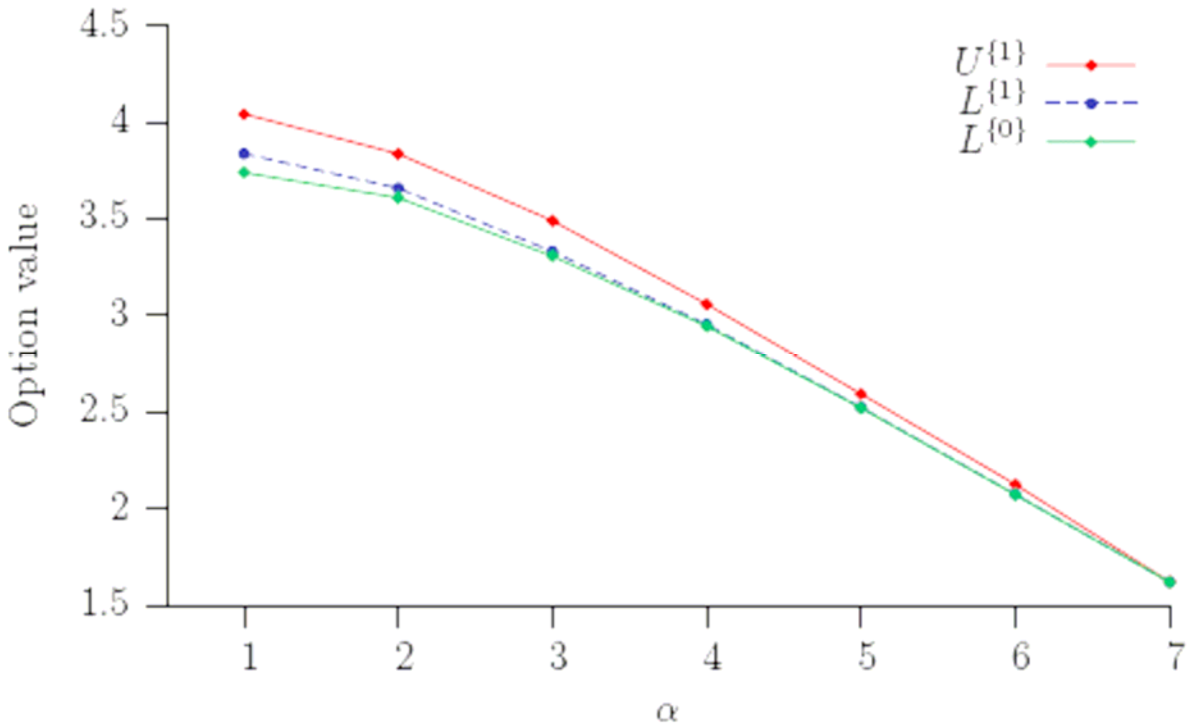
standard: $$A_{std} = U_\alpha - L_\alpha$$

new (estim.): $$A_{est} = L_\alpha^{\{2\}} - L_\alpha^{\{1\}}$$

α	$L_\alpha^{\{2\}}(\%)$	$A_{std}^{\{2\}}$ (bp)	$A_{est}^{\{2\}}$ (bp)
1	3.977	6	6
2	3.977	5	6
3	3.952	4	5
4	3.891	4	4
5	3.796	3	4
6	3.675	1	2
7	3.516	0	2

(*)1 bp = 0.01 %

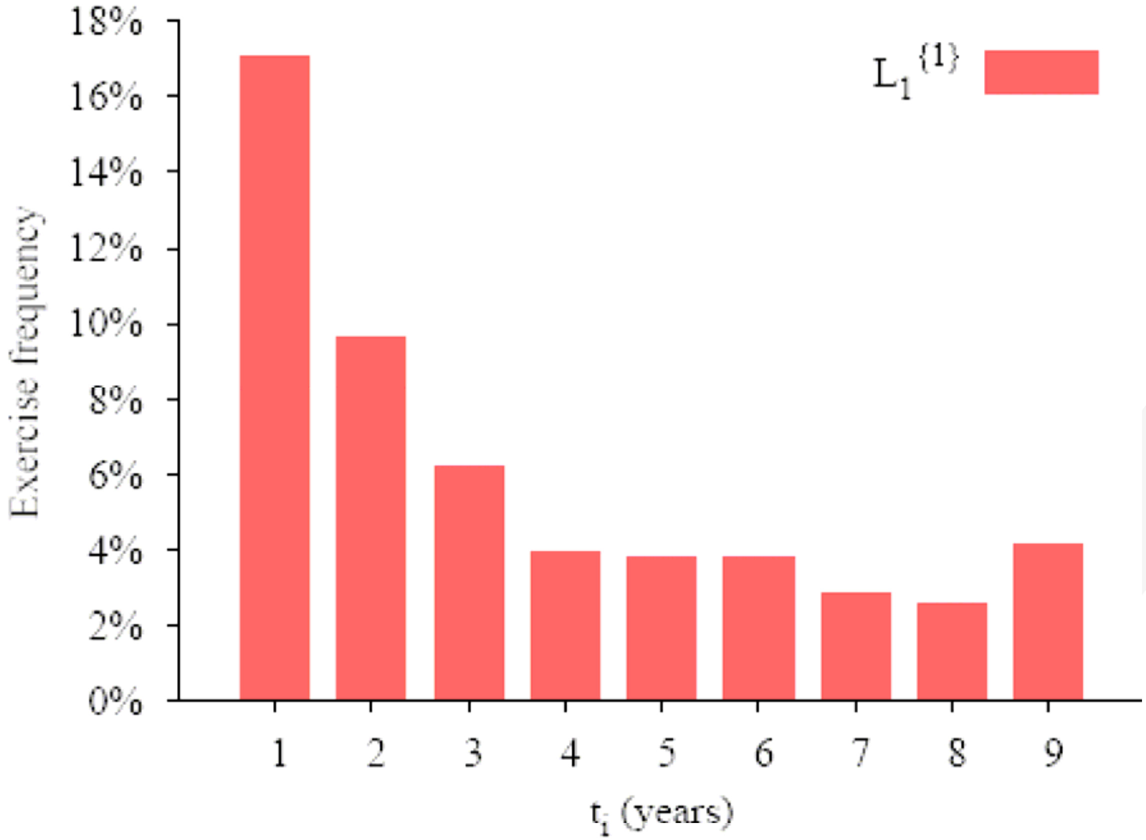
Example 2: CMS Spread Bermudan



L_α, U_α # paths as before…

Payoff: 5 (CMS_{10} – CMS_2), floored @ 0.5% capped @ 8%

Example 2: Exercise Frequency



Conclusions

Bond Market Model :

- A simple multi-factor model
- Black like formulas for caps/floors and swaptions
- MonteCarlo: Markov between reset dates

Gigi model :

- An elementary (multi-factor) model with smile
- Simple calibration: 1-d closed formulas for PV
- Straightforward MC

Callable Exotics :

- High precision
- Fast (no maximization)
- Accuracy control

Bibliography sketch

L.B.G. Andersen (2000), *A Simple Approach to the Pricing of Bermudan Swaptions in the Multi-Factor Libor Market Model,* J. Computational Finance 3, 1-32

L.B.G. Andersen & M. Broadie (2004), *A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options*, Management Science 50, 1222-1234

R. Baviera (2006), *Bond Market Model*, IJTAF 9, 577-596

R. Baviera and L. Giada (2006), *A perturbative approach to Bermudan Option pricing with applications*, Preprint, Available at www.ibleo.it

R. Baviera (2007), *Gigi Model: a simple stochastic volatility approach for multifactor interest rates*, Preprint, Available at SSRN: http://ssrn.com/abstract=977479

F. Black (1976), *The pricing of commodity contracts*, J. Financial Economics 3, 167-179

A. Brace, D. Gatarek and M. Musiela (1997), *The Market Model of Interest Rate Dynamics*, Mathematical Finance 7, 127-155

Bibliography sketch

D. Brigo and F. Mercurio, *Interest Rate Models: Theory and Practice*, Springer

D. Brigo, F. Mercurio and F. Rapisarda (2004), *Smile at the Uncertainty*, Risk 5, 97-101

J. Gatheral (2006), *The Volatility Surface: A Practitioner's Guide*, Wiley.

P. Glasserman (2003), *Monte Carlo Methods in Financial Engineering*, Springer

D. Heath, R. Jarrow and A. Morton (1992), *Bond Pricing and the Term Structure of Interest Rates: a New Methodology for Contingent Claims Valuation*, Econometrica 60, 77-105

A. Lewis (2001), *A Simple Option Formula for General Jump-Diffusion and Other Exponential Levy Processes*. Available at SSRN: http://ssrn.com/abstract=282110

F. Mercurio and A. Pallavicini (2006), *Smiling at convexity: bridging swaption skews and CMS adjustments*, Risk 8, 64-69

M. Musiela and M. Rutkowsky (1997), *Martingale Methods in Financial Modeling*, Springer