A perturbative approach to Bermudan Options pricing with applications

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Callable products: Problem Formulation

Bermudan option:

$$C_0 = \sup_{\tau \in \mathcal{T}} E_0 \left[D_{0\tau} \, h \left(\tau, \mathbf{B}(\tau) \right) \right]$$

$\mathcal{T}$: class of admissible stopping times with values in $\{t_1, t_2, \ldots, t_N\}$

Optimal stopping τ^*

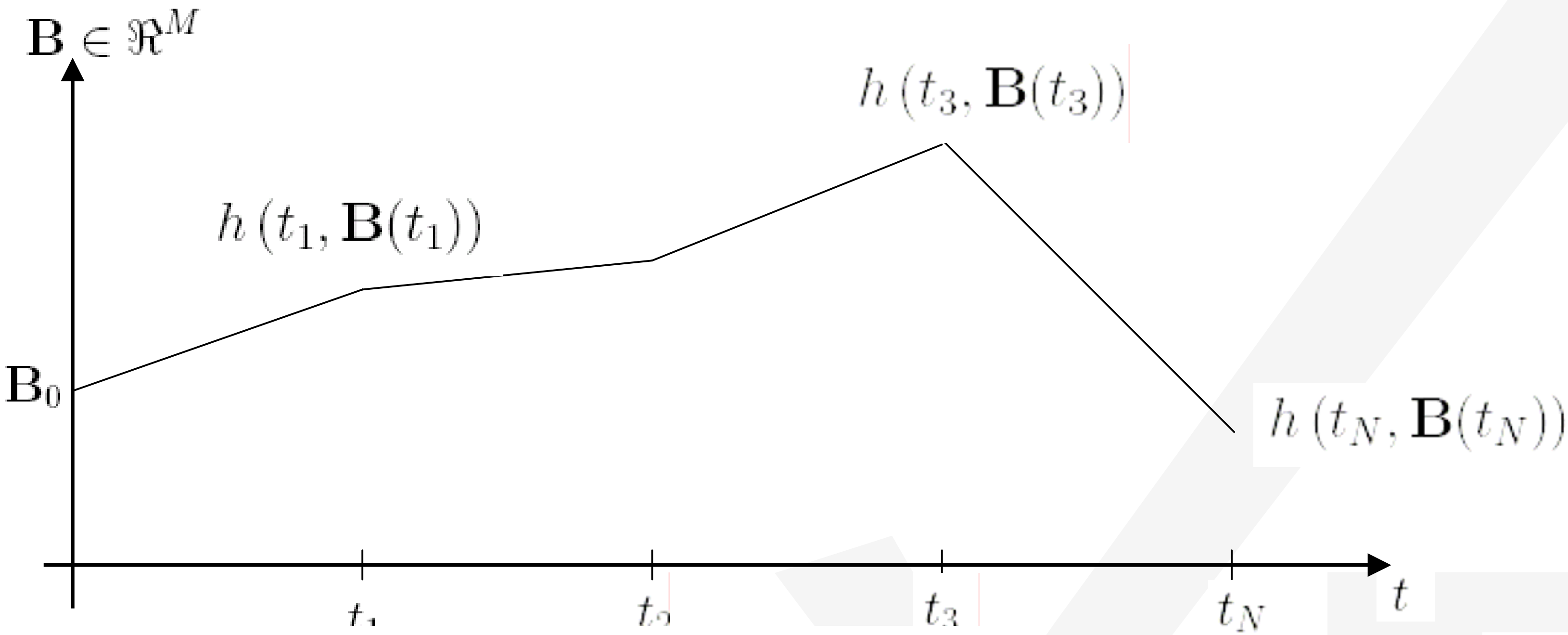
$$\tau^* = \min_i \left\{ t_i : h\left(t_i, \mathbf{B}(t_i)\right) \geq C_i\left(\mathbf{B}(t_i)\right) \right\}$$

with $C_i(\mathbf{B})$: Continuation value function

$D_{0\,i}$: discount in (t_0, t_i)

$h\left(t_i, \mathbf{B}(t_i)\right)$: payoff in t_i

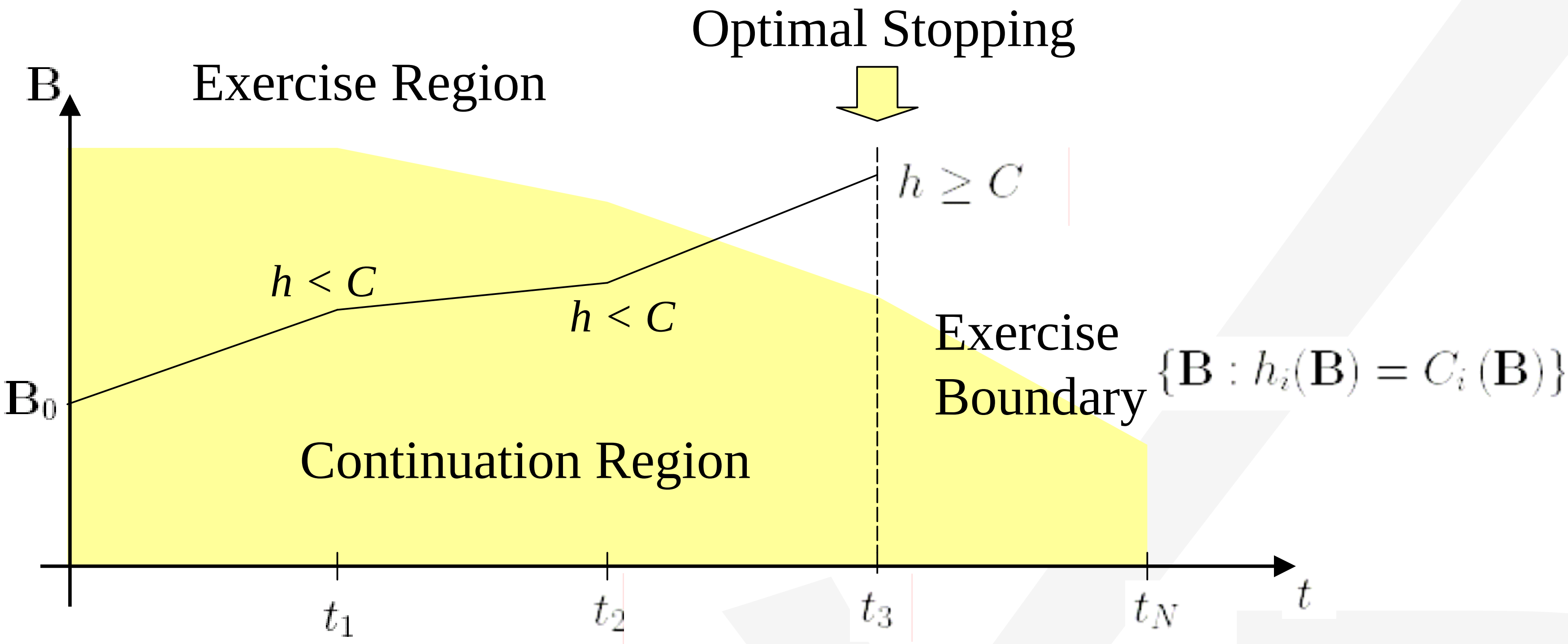
MonteCarlo: std approach for Non-Callable products



$$O = E\left[\sum_i D_{0\,i} h(t_i, \mathbf{B}(t_i))\right]$$

Why MonteCarlo? Lattice methods work poorly for high-dimentional problems.

Callable products: MonteCarlo approach



Problem:

$C_i(\mathbf{B})$ is a Bermudan option with exercise dates $\{t_{i+1}, t_{i+2}, \ldots, t_N\}$

In a MC approach each $C_i(\mathbf{B})$ should come from a **new** MC simulation starting in t_i !?!

Any approximate exercise strategy $\hat{\tau}$ provides a lower bound

$$L_0 = E_0\left[D_{0\hat{\tau}}\, h\left(\hat{\tau}, \mathbf{B}(\hat{\tau})\right)\right] \leq E_0\left[D_{0\tau^*}\, h\left(\tau^*, \mathbf{B}(\tau^*)\right)\right] = C_0$$

using in the exercise decision an approximation $\hat{C}_i\left(\mathbf{B}, \{\eta\}\right)$

where $\{\eta\}$ are a set of parameters…

Idea:

Option value *not very sensitive* to the **exact** position of the Exercise Boundary

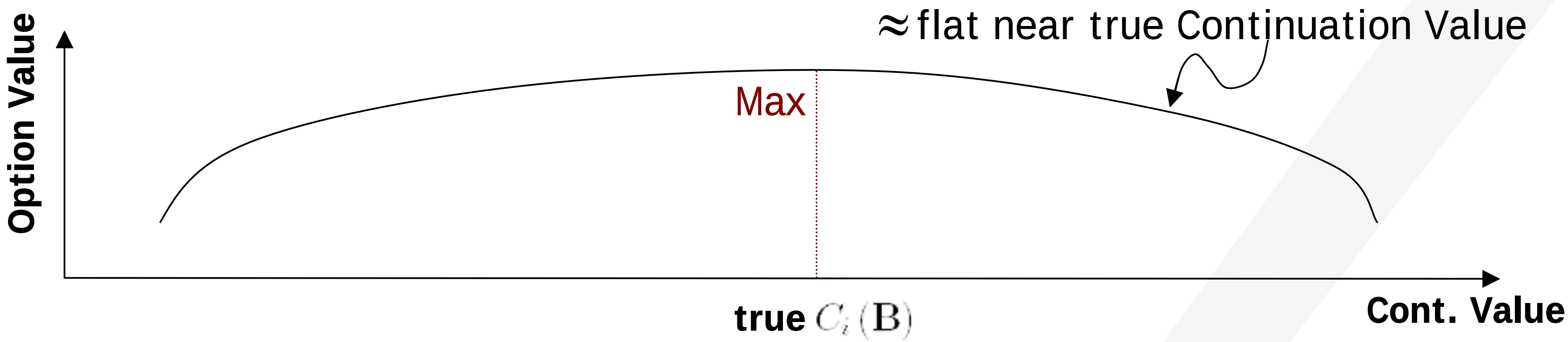
Even a rough approximation of $C_i\left(\mathbf{B}\right)$ leads to a reasonable approximation of option value

Two standard approaches:

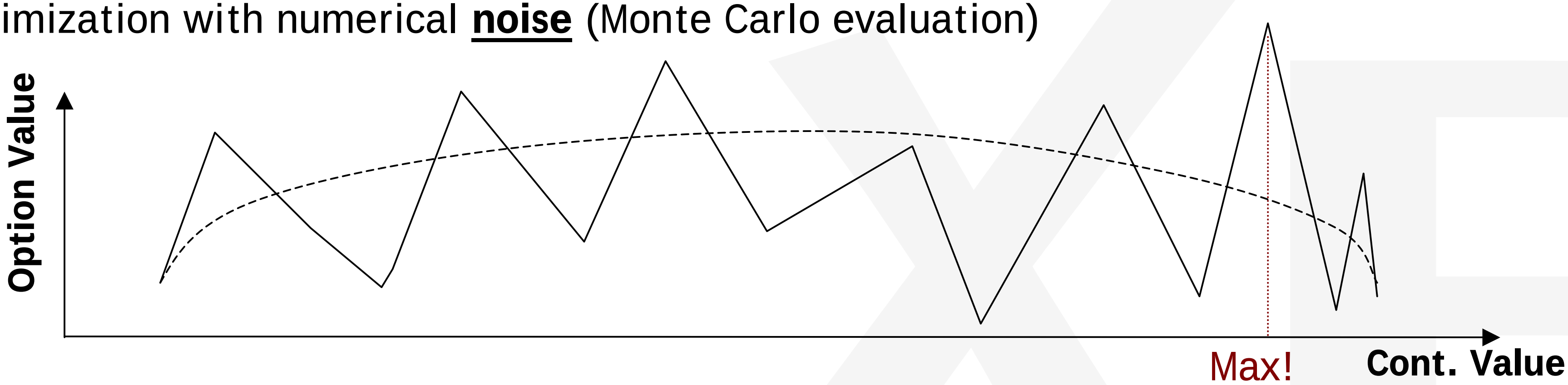
- Longstaff-Schwartz (1998)
- Andersen (2000)

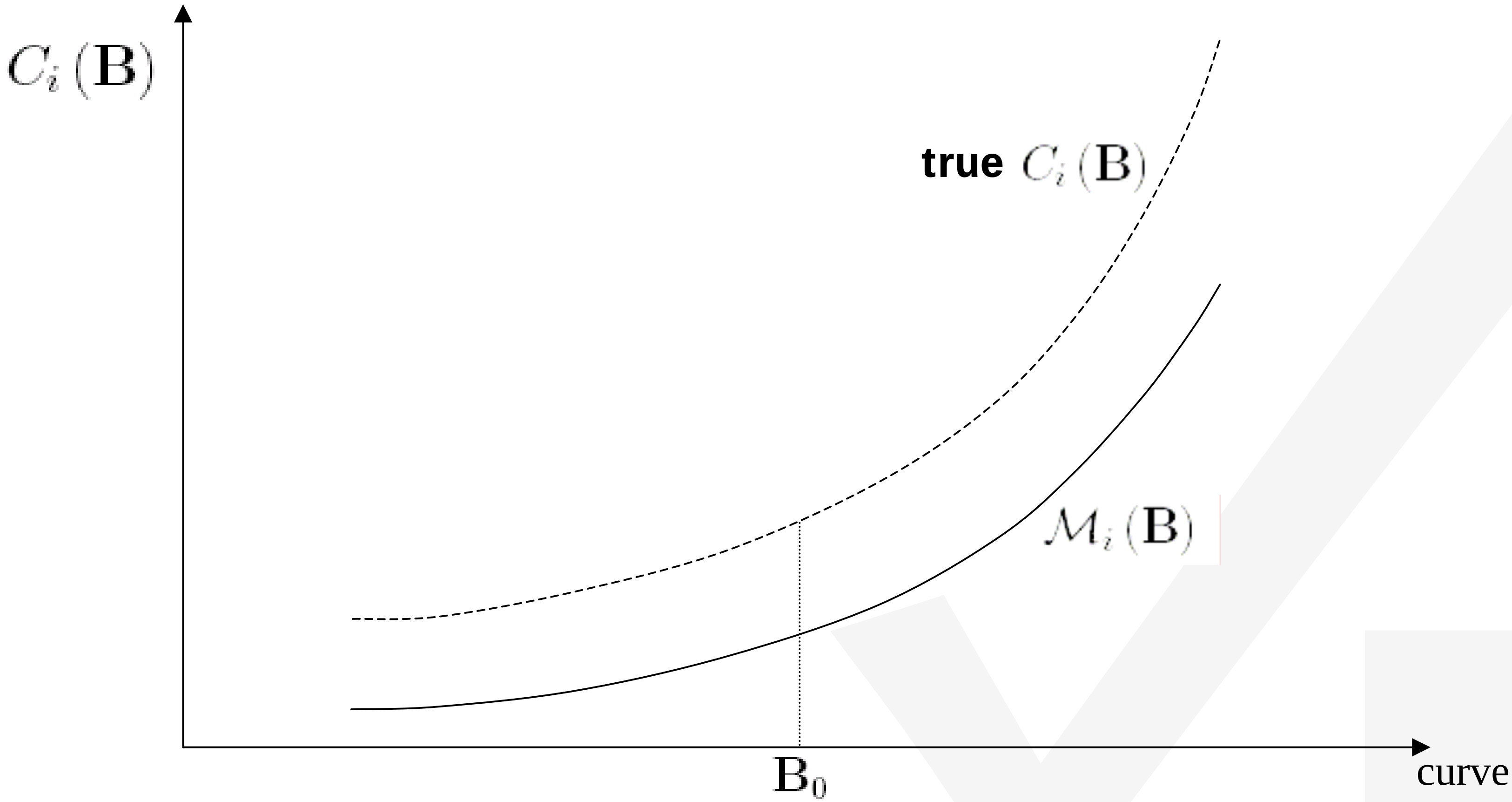
…then find the best $\{\eta\}$. (Andersen 2000)

Optimization **exact** function

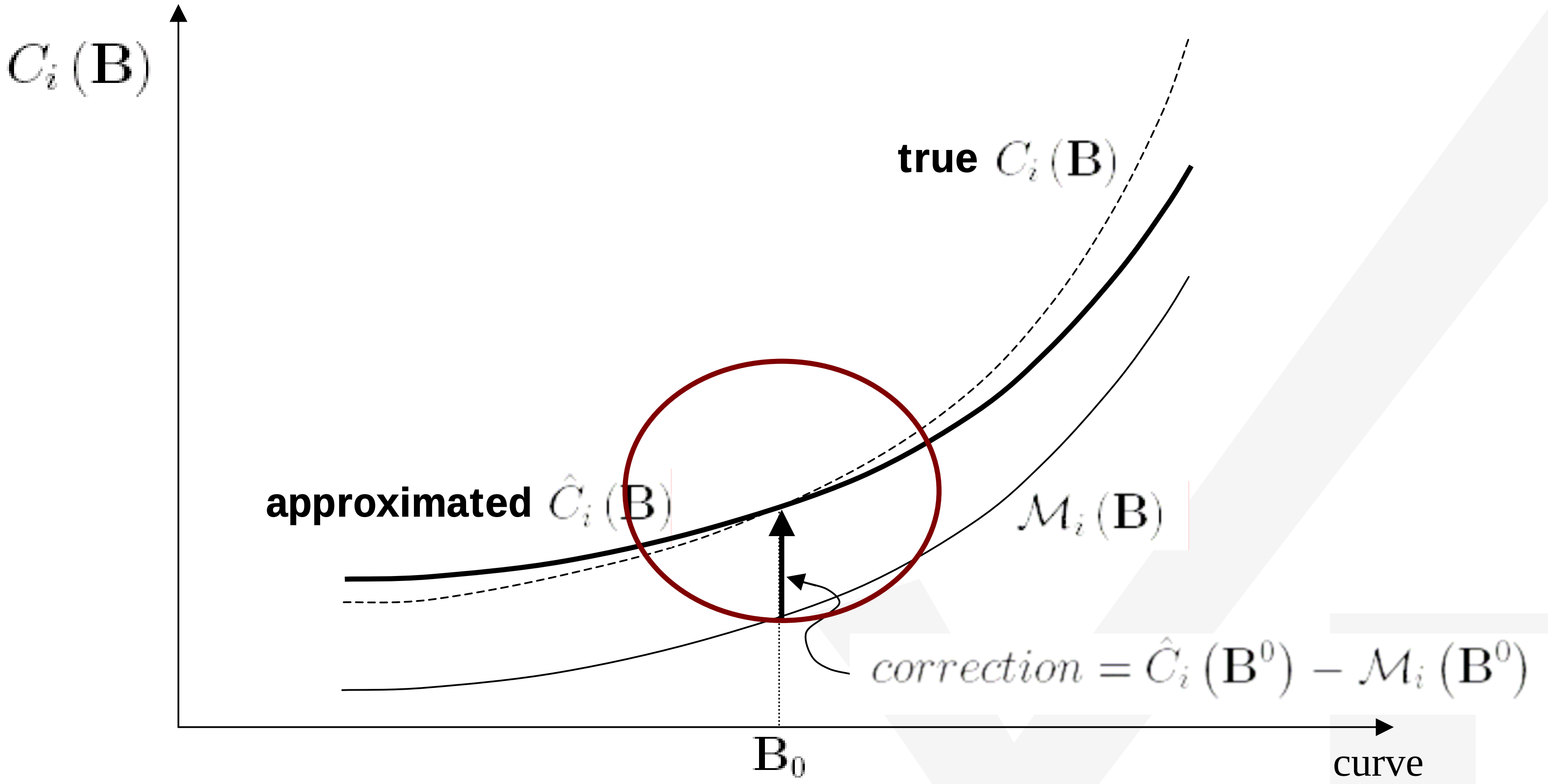


Optimization with numerical **noise** (Monte Carlo evaluation)





with $\mathcal{M}_i(\mathbf{B})$ an arbitrary **simple** (to compute) function



$$\hat{C}_i(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + correction$$

Starting from the (N-1) Continuation value function, already a simple function,

how to get $\hat{C}_i(\mathbf{B})$ knowing $\hat{C}_n(\mathbf{B}) \quad \forall n > i$

$$\hat{C}_n(\mathbf{B}) \quad \forall n > i$$

⇓

$$\hat{C}_i(\mathbf{B}^0)$$

⇓

$$\hat{C}_i(\mathbf{B}) \equiv \mathcal{M}_i(\mathbf{B}) + \left(\hat{C}_i(\mathbf{B}_0) - \mathcal{M}_i(\mathbf{B}_0)\right)$$

$\mathcal{M}_i(\mathbf{B})$ a possible choice

$$\mathcal{M}_i(\mathbf{B}) \equiv \mathcal{E}_m(\mathbf{B}, t_i) \qquad 0 < i < m \leq N$$

with $\mathcal{E}_m$ the max European option in $\mathbf{B}^0$: $\mathcal{E}_m(\mathbf{B}^0, t_i) \geq \mathcal{E}_n(\mathbf{B}^0, t_i) \ \forall n$

where $\mathcal{E}_n(\mathbf{B}^0, t_i)$ European option valued in t_i with expiry t_n

$$\hat{C}_i^{\{1\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{1\}}(i)$$

$$\hat{C}_i^{\{2\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{2\}}(i) + \sum_{n=i}^{N} c_1^{\{2\}}(i,n)\left(\ln B_n - \ln B_n^0\right)$$

...

$$c_0^{\{1\}}(i) = \left[\hat{C}_i^{\{1\}} - \mathcal{M}_i\right](\mathbf{B}^0)$$

value in $\mathbf{B}^0$

$$c_1^{\{2\}}(i,n) = B_n^0 \frac{\partial}{\partial B_n}\left[\hat{C}_i^{\{2\}} - \mathcal{M}_i\right](\mathbf{B}^0)$$

Delta in $\mathbf{B}^0$

...

$$c_2^{\{3\}}(i,j,n) = \cdots$$

Gamma in $\mathbf{B}^0$

Idea:

Given Π a class of martingale processes with values in $\{t_1, t_2, \ldots, t_N\}$

$$\sup_{\tau \in \mathcal{T}} E_0 \left[D_{0\tau}\, h\left(\tau, \mathbf{B}(\tau)\right)\right] = C_0 = \inf_{\pi \in \Pi} \left\{ \pi_0 + E_0 \left[\max_i \left(D_{0i} h_i - \pi_i \right) \right] \right\}$$

Lower Bound: L_0

Upper Bound: U_0

(Roger 2001, Andersen Broadie 2004, ...)

An approximated continuation value function set $\left\{\hat{C}_i(\mathbf{B})\right\}_i$

⇩

martingale process $\{\hat{\pi}_i\}_i$

$$\begin{cases} \hat{\pi}_0 = L_0 \\ \hat{\pi}_i = \hat{\pi}_{i-1} + \Delta\pi_i \quad i = 1,..,N \end{cases}$$

with:

$$\begin{aligned} \Delta\pi_i &= D_{0i+1}\left\{\max\left(h_{i+1}, \hat{C}_{i+1}\right) - E_i\left[\max\left(h_{i+1}, \hat{C}_{i+1}\right)\right]\right\} && i = 1, ..., N-1 \\ \Delta\pi_N &= D_{0N}h_N - D_{0N-1}\hat{C}_{N-1} && i = N \end{aligned}$$

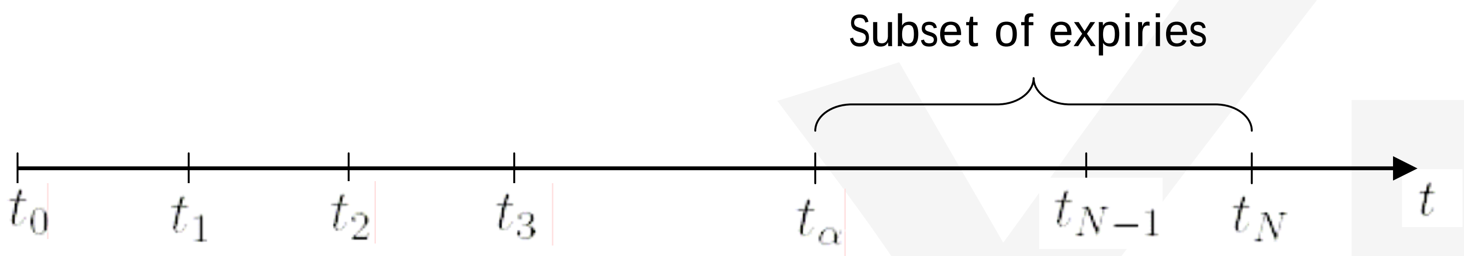
...two nested MCs

Examples

1. 10y S/A ZC Bermudan option (N = 19)
2. 10y S/A Step Up Callable (N = 19)
3. 10y A/A Bermudan option on a 10-2 CMS spread (N = 9)

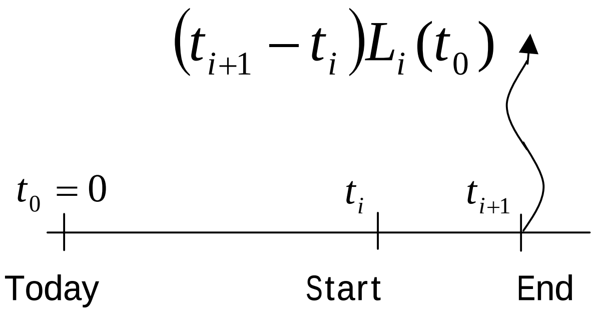
We also consider Lower and Upper bounds for Bermudans with a subset of exercise dates

L_α , U_α : first expiry in t_α

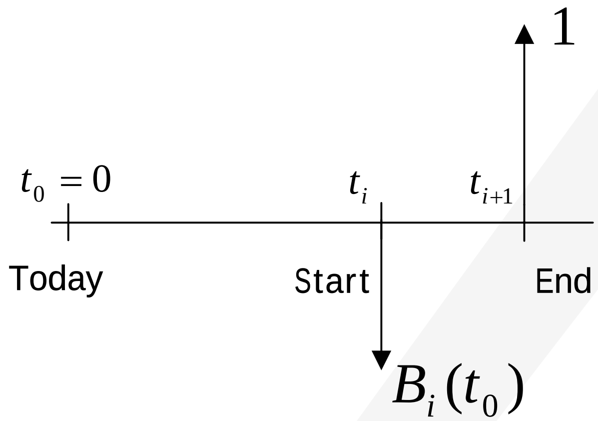


Model: Notation

Forward Libor Rates (in t_0) $L_i(t_0)$



Forward ZC Bond (in t_0) $B_i(t_0)$



... and their relation

$$L_i(t_0) = \frac{1}{t_{i+1} - t_i} \left\{ \frac{1}{B_i(t_0)} - 1 \right\}$$

BMM Dynamics: spot measure

$$dB_i(t) = B_i(t) v_i \left\{ - \sum_{j=k+1}^{i} \rho_{ij}^{(B)} v_j \, dt + dW(t) \right\}$$

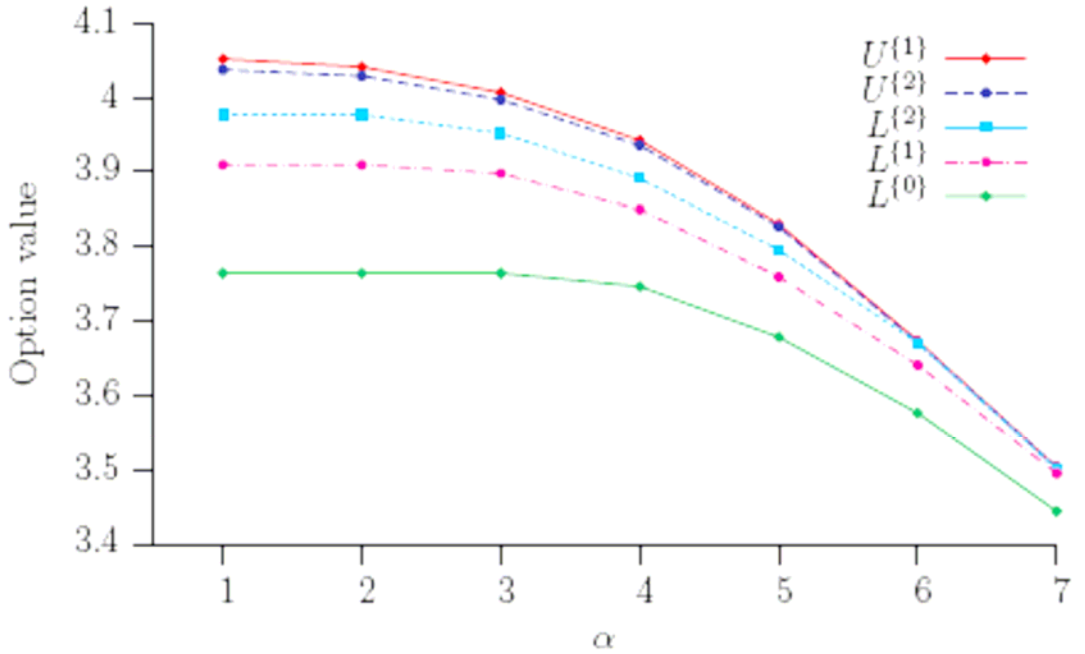
$$\text{with} \quad t_k \le t < t_{k+1} \quad ; \quad dW_i(t) \bullet dW_j(t) = \rho_{ij}^{(B)} dt$$

$v_i = 0 \quad for \quad t \ge t_i$ ⇨ Fixing Mechanism

Some BMM Advantages

- Elementary MC: Markov between Reset dates (Gaussian HJM)
- Black like formulas for Caps/Floors & Swaptions
- Large set of analytical solutions (e.g. CMS & CMS Spread European Options) …

Example 1: ZC Bermudan Option



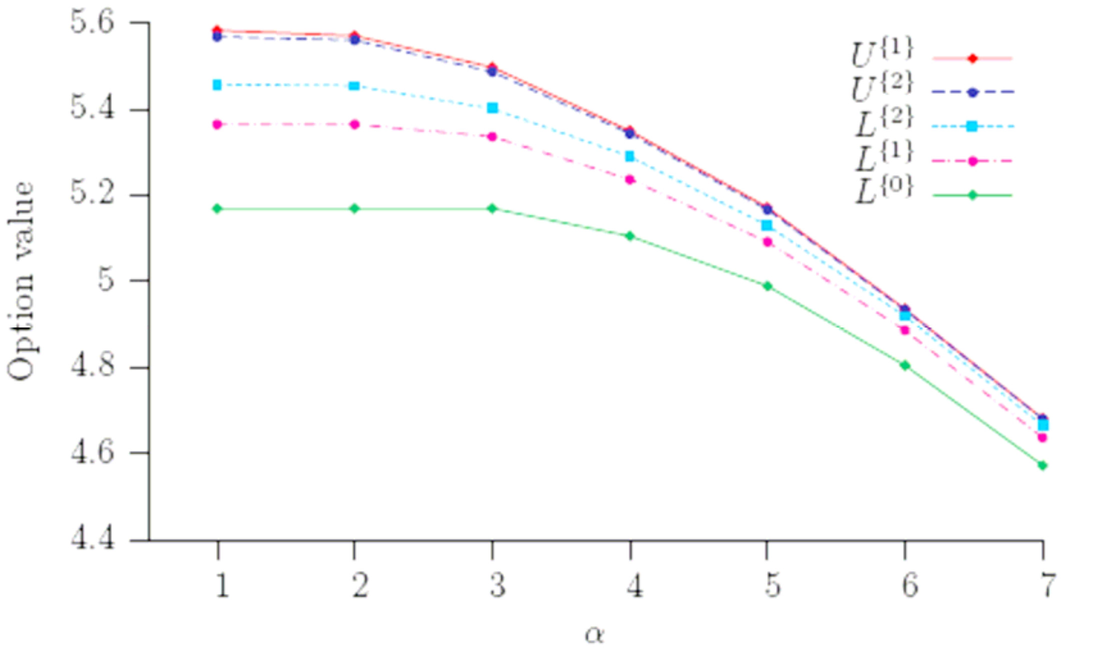
L_α using 10^6 paths

U_α using $5 \cdot 10^4$ paths (external MC) & 10^3 paths (internal MC)

Strikes (N=19): $K_i = 8.75 \cdot 10^{-4}\, i^2 + 1.09 \cdot 10^{-2}\, i + 0.432$

Dataset: 14 Jan 05 at 11:15 CET

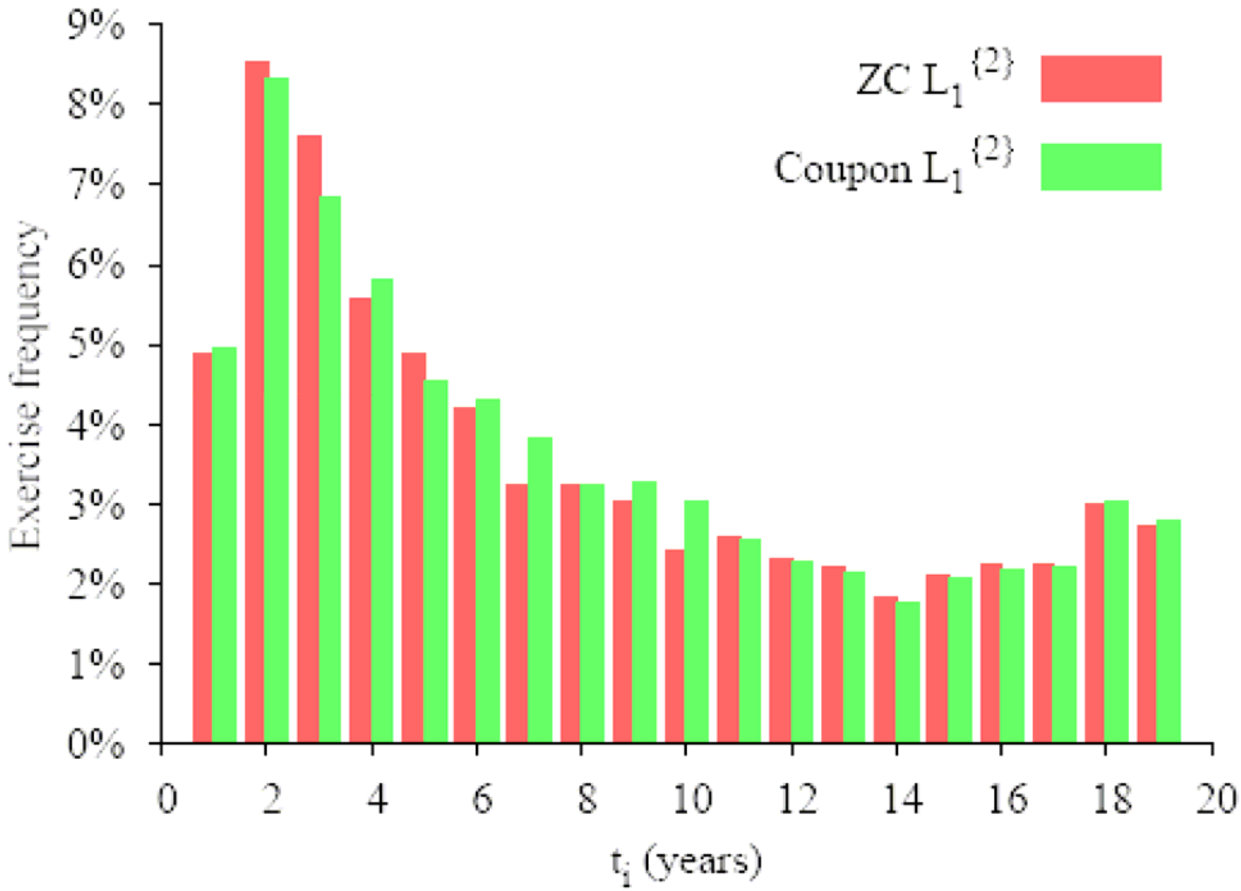
Example 2: Bermudan Coupon Option



L_α, U_α # paths as before…

10y S/A Stepped Up yearly by 0.2% (2.9 % - 4.7 %)

Exercise Frequency



New Approach: Accuracy

Option value $$C_\alpha = L_\alpha + \frac{A}{2}$$

Accuracy in bps(*)

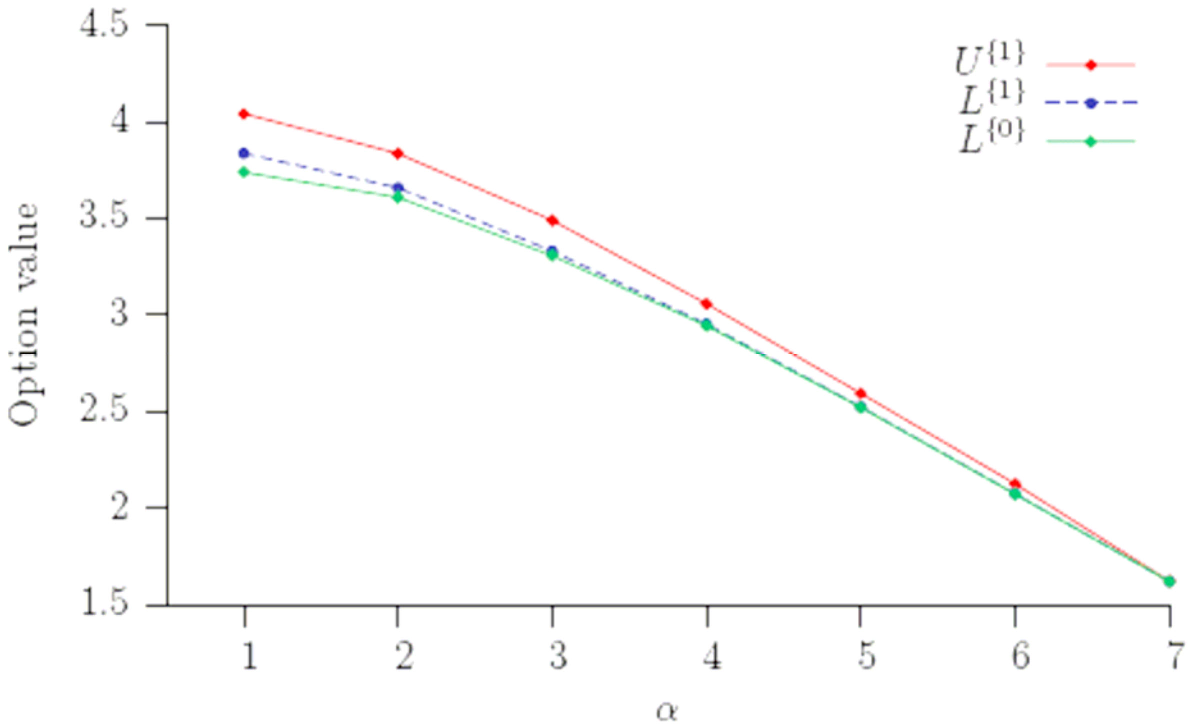
standard: $$A_{std} = U_\alpha - L_\alpha$$

new (estim.): $$A_{est} = L_\alpha^{\{2\}} - L_\alpha^{\{1\}}$$

α	$L_\alpha^{\{2\}}$ (%)	$A_{std}^{\{2\}}$ (bp)	$A_{est}^{\{2\}}$ (bp)
1	3.977	6	6
2	3.977	5	6
3	3.952	4	5
4	3.891	4	4
5	3.796	3	4
6	3.675	1	2
7	3.516	0	2

(*)1 bp = 0.01 %

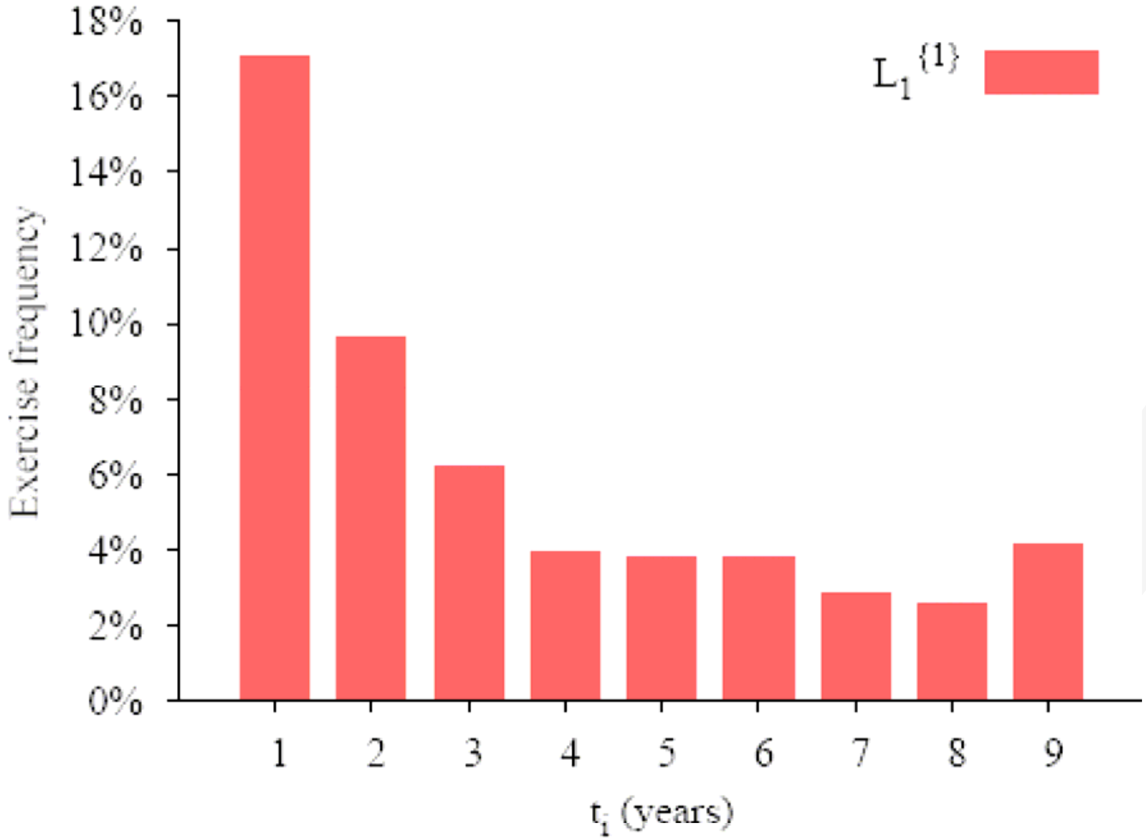
Example 3: CMS Spread Bermudan



L_α, U_α # paths as before…

Payoff: 5 (CMS_{10} – CMS_2), floored @ 0.5% capped @ 8%

Example3: Exercise Frequency



Conclusions

An elementary new tecnique for pricing Bermudans with Multi-factor models:

- Methodology is model independent …
- … and related to financial quantities
- High precision
- Fast (no maximization)
- Accuracy control

Bibliography sketch

L.B.G. Andersen (2000), *A Simple Approach to the Pricing of Bermudan Swaptions in the Multi-Factor Libor Market Model,* J. Computational Finance 3, 1-32

L.B.G. Andersen & M. Broadie (2004), *A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options*, Management Science 50, 1222-1234

R. Baviera (2006), *Bond Market Model*, IJTAF 9, 577-596

R. Baviera and L. Giada (2006), *A perturbative approach to Bermudan Option pricing*, http://ssrn.com/abstract=941318 & http://www.ibleo.it

P. Glasserman (2003), *Monte Carlo Methods in Financial Engineering*, Springer

D. Heath, R. Jarrow and A. Morton (1992), *Bond Pricing and the Term Structure of Interest Rates: a New Methodology for Contingent Claims Valuation*, Econometrica 60, 77-105

F. Longstaff, E. Schwartz (1998), *Valuing American options by simulation: a least squares approach*, Rev. Fin. Studies 14,113–147

C. Rogers (2002), *Monte Carlo Valuation of American Options,* Mathematical Fin. 12, 271-286