Bond Market Model
Its Implementation
Uses for Pricing Callable Exotics

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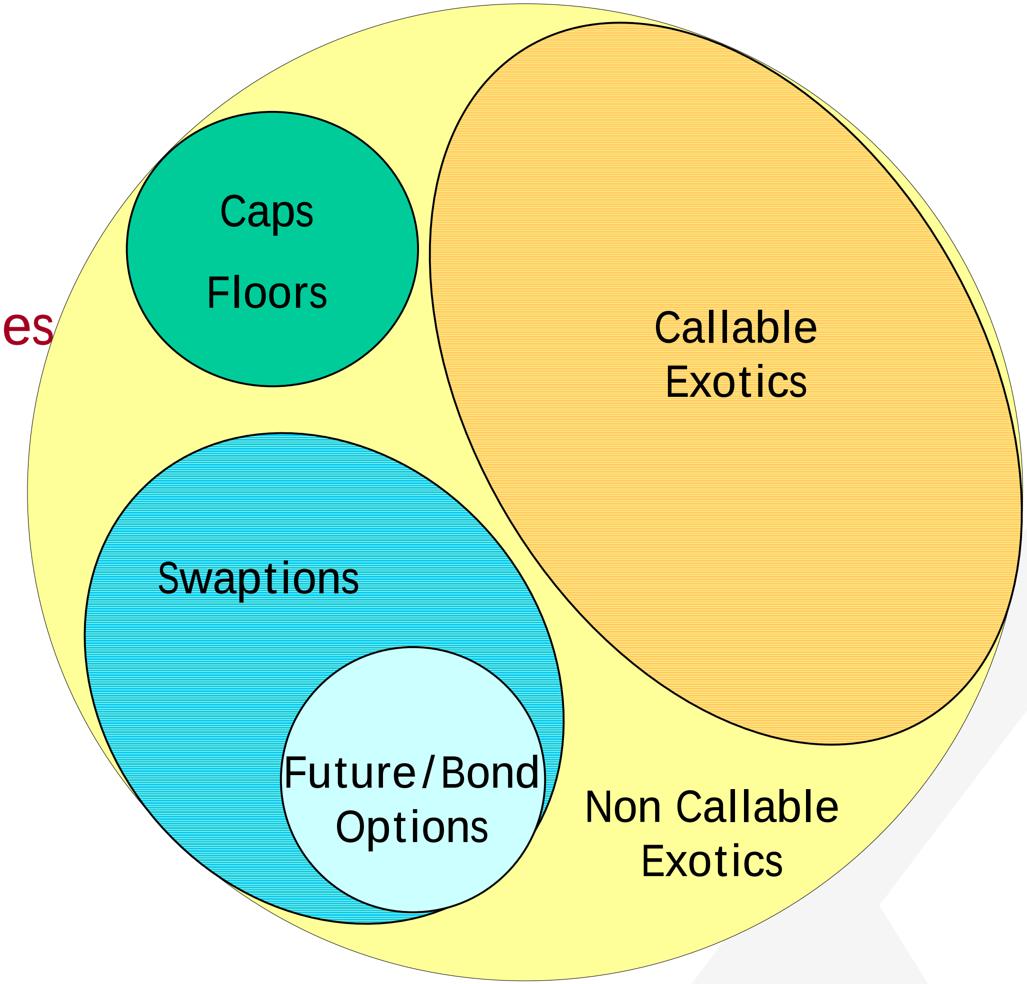
London, 8 Dec 2006

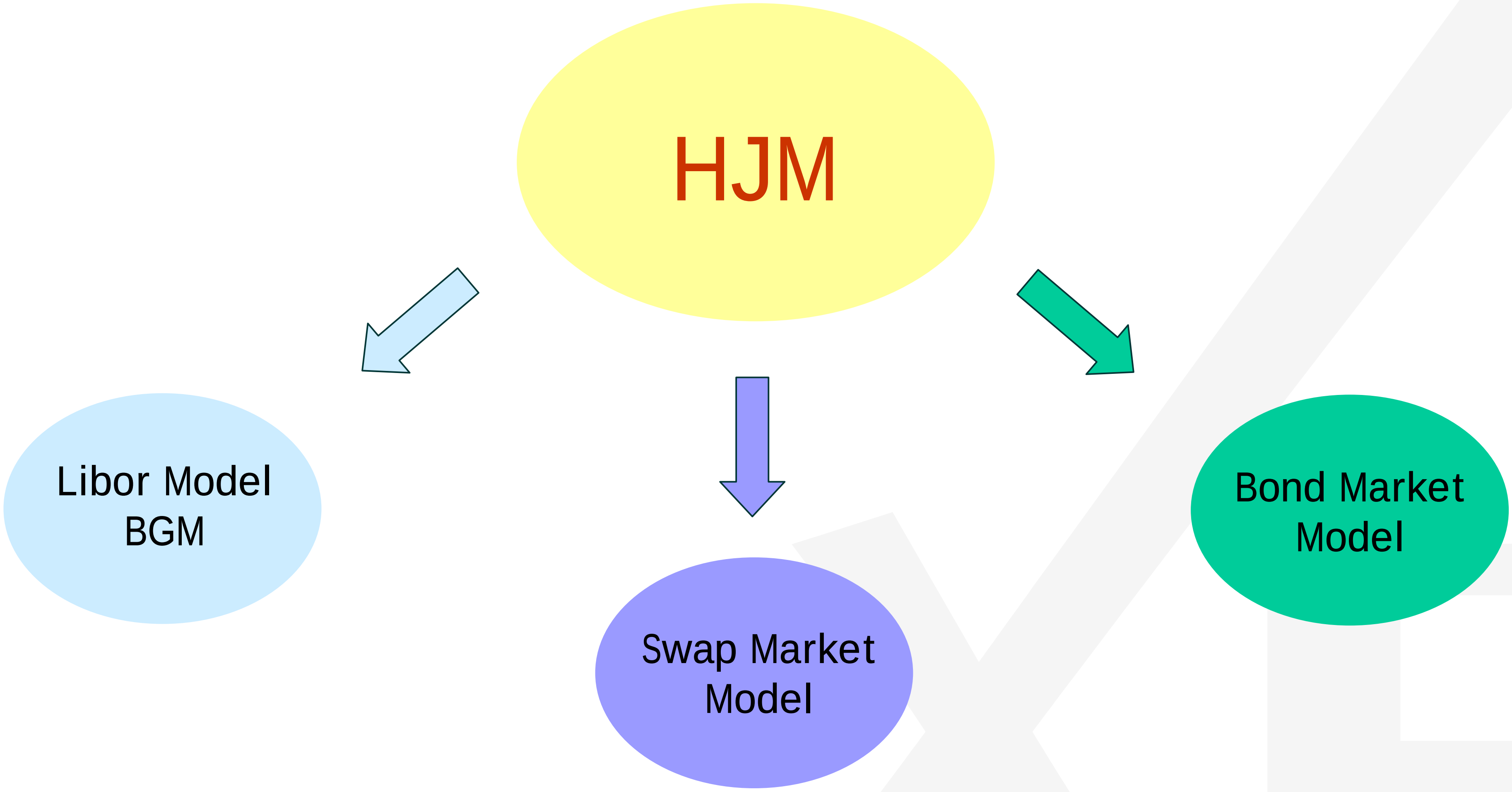
Outline

- Overview: IR markets, products & models
- Bond Market Model
  - vs Libor Market Model
  - Cap/Floor & Swaption Formulas
  - Monte Carlo simulations and security valuation
- Callable Exotics
  - Formulation
  - Lower Bound: Standard Approach
  - Lower Bound: Perturbation Approach
  - Upper Bound
  - Putting Numbers In

Interest Rates Market: Products

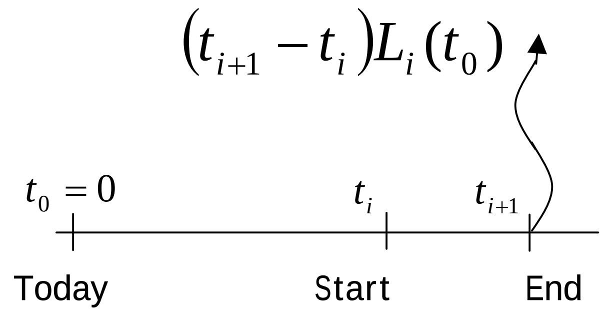
Interest Rates Derivatives



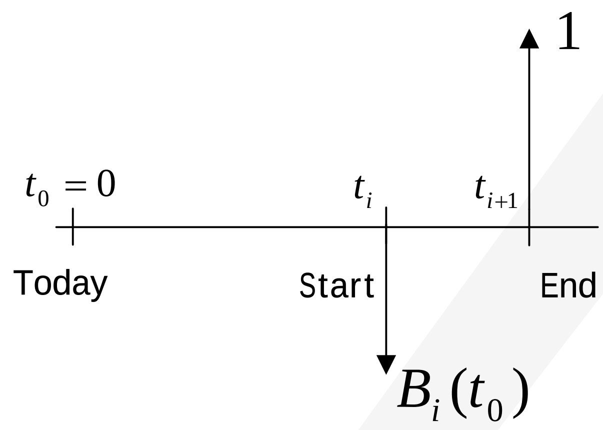

HJM
Libor Model
BGM
Swap Market
Model
Bond Market
Model

IR Market: Libor Rates & Discount Factors

Forward Libor Rates (in t_0) $L_i(t_0)$



Forward ZC Bond (in t_0) $B_i(t_0)$

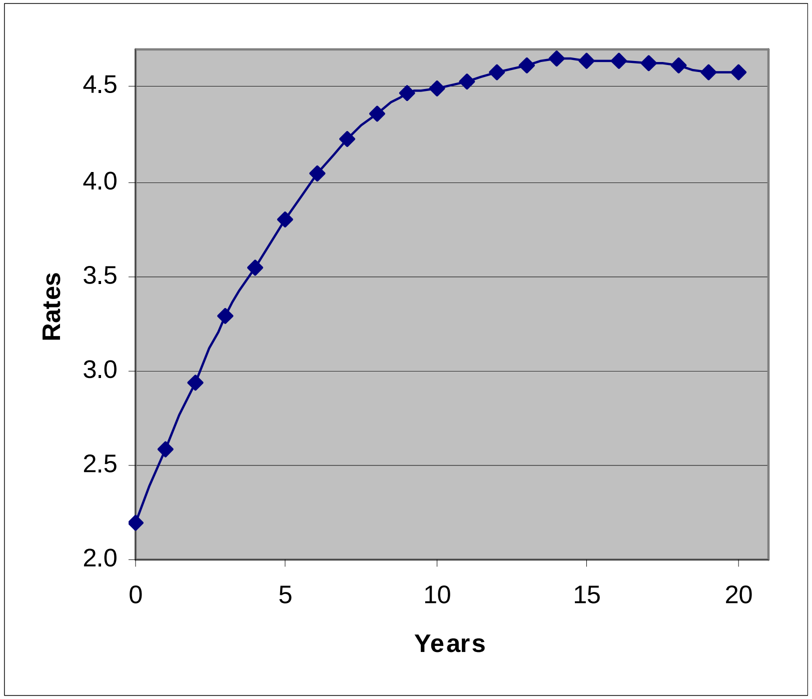


… and their relation

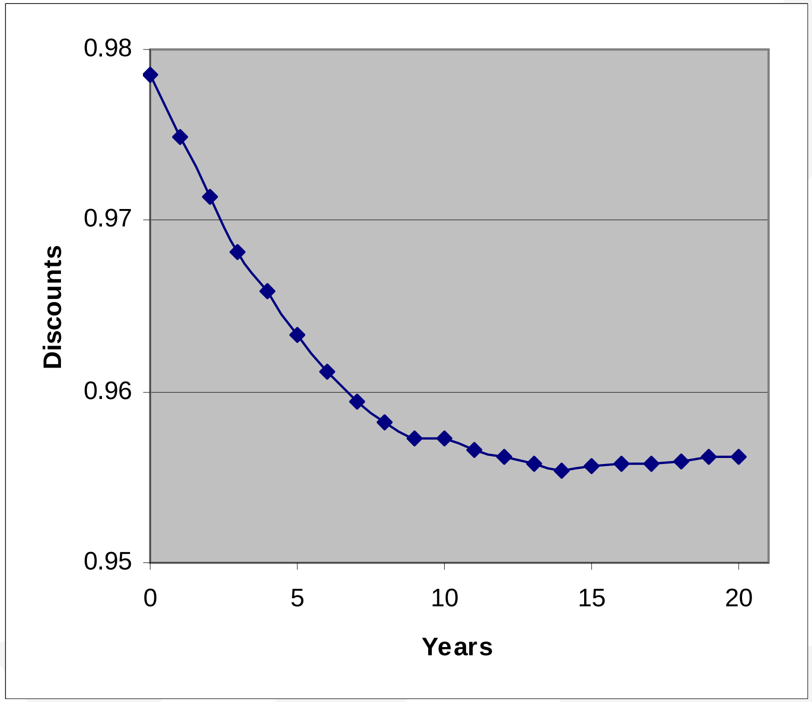
$$L_i(t_0) = \frac{1}{t_{i+1} - t_i} \left\{ \frac{1}{B_i(t_0)} - 1 \right\}$$

…

Rates $L_i(T_0)$



Bonds $B_i(T_0)$



A possible model…

$$dL_i(t) = (\cdots)dt - L_i(t)\sigma_i dW(t)$$

… and another possibility

$$dB_i(t) = (\cdots)dt + B_i(t)v_i dW(t)$$

Dataset: 14 Jan 05 at 11:15 CET

LMM Dynamics: spot measure

$$dL_i(t) = L_i(t)\sigma_i \left\{ \sum_{j=k+1}^{i} \rho_{ij}^{(L)} \sigma_j \frac{(t_{j+1} - t_j) L_j(t)}{1 + (t_{j+1} - t_j) L_j(t)} dt - dW(t) \right\}$$

with $t_k \leq t < t_{k+1}$; $dW_i(t) \bullet dW_j(t) = \rho_{ij}^{(L)} dt$

$\sigma_i = 0 \quad for \quad t \geq t_i$ ⇨ Fixing Mechanism

LMM Dynamics: t_{i+1} forward measure

$$dL_i(t) = -L_i(t)\sigma_i dW^{(i+1)}(t)$$

BMM Dynamics: spot measure

$$dB_i(t) = B_i(t) v_i \left\{ - \sum_{j=k+1}^{i} \rho_{ij}^{(B)} v_j \, dt + dW(t) \right\}$$

$$\text{with} \quad t_k \leq t < t_{k+1} \quad ; \quad dW_i(t) \bullet dW_j(t) = \rho_{ij}^{(B)} dt$$

$v_i = 0 \quad for \quad t \geq t_i$ ⇒ Fixing Mechanism

BMM Dynamics: t_i forward measure

$$dB_i(t) = B_i(t) v_i \, dW^{(i)}(t)$$

Models: Caplet (Exact) Solutions

Libor Model

$$Caplet_i = B_{i+1}(t_0)\left(t_{i+1} - t_i\right)\left\{L_i(t_0) \bullet N(d_+^{cf}) - K \bullet N(d_-^{cf})\right\}$$

$$d_+^{cf} = \frac{1}{\sigma_i \sqrt{t_i}} \ln\left(\frac{L_i(t_0)}{K}\right) + \frac{1}{2}\sigma_i \sqrt{t_i} \quad ; \quad d_-^{cf} = d_+^{cf} - \sigma_i \sqrt{t_i}$$

Bond Model

$$Caplet_i = B_{i+1}(t_0)\left\{\left[1 + \left(t_{i+1} - t_i\right)L_i(t_0)\right] \bullet N(d_+^{(B)}) - \left[1 + \left(t_{i+1} - t_i\right)K\right] \bullet N(d_-^{(B)})\right\}$$

$$d_+^{(B)} = \frac{1}{v_i \sqrt{t_i}} \ln\left(\frac{1 + \left(t_{i+1} - t_i\right)L_i(t_0)}{1 + \left(t_{i+1} - t_i\right)K}\right) + \frac{1}{2} v_i \sqrt{t_i} \quad ; \quad d_-^{(B)} = d_+^{(B)} - v_i \sqrt{t_i}$$

Libor Model

- Exact: *Not Available*
- Approximated: $SP_{\alpha\omega} = B_{0\alpha}(T_0)BPV_{\alpha\omega}(T_0)\left\{S_{\alpha\omega}(T_0)\bullet N(d_1^s) - k\bullet N(d_2^s)\right\}$

$$d_1^s = \frac{1}{\sigma_{\alpha\omega}\sqrt{T_\alpha}}\ln\left(\frac{S_{\alpha\omega}(T_0)}{k}\right) + \frac{1}{2}\sigma_{\alpha\omega}\sqrt{T_\alpha}; \qquad d_2^s = d_1^s - \sigma_{\alpha\omega}\sqrt{T_\alpha};$$

$$\sigma_{\alpha\omega}^2 = \sum_{i,j=\alpha}^{\omega-1}\eta_i\sigma_i\rho_{ij}^{(L)}\eta_j\sigma_j$$

Not so good…

Bond Model

- Exact: *Available* (see e.g. Musiela Rutkowski 1997)
- Approximated: $SP_{\alpha\omega} = B_{0\alpha}(T_0)\left\{N(-d_2^B) - P_{\alpha\omega}(k;T_0)\bullet N(-d_1^B)\right\}$

$$d_1^B = \frac{1}{V_{\alpha\omega}\sqrt{T_\alpha}}\ln\left(P_{\alpha\omega}(k;T_0)\right) + \frac{1}{2}V_{\alpha\omega}\sqrt{T_\alpha}; \qquad d_2^B = d_1^B - V_{\alpha\omega}\sqrt{T_\alpha}$$

$$V_{\alpha\omega}^2 = \sum_{i=\alpha+1}^{\omega}\gamma_i v_i\rho_{ij}^{(B)}\gamma_i v_i \quad \& \quad P_{\alpha\omega}(k;T_0) = k\sum_{i=\alpha+1}^{\omega}B_{\alpha i}(T_0) + B_{\alpha\omega}(T_0)$$

Precision… let we see

Approximation Precision of the Bond Market solution

Our goal is to measure only the error
coming from the *approximation* of swaption formula

We plot the difference between the exact and the approximated solution
in the Euro market with a correlation

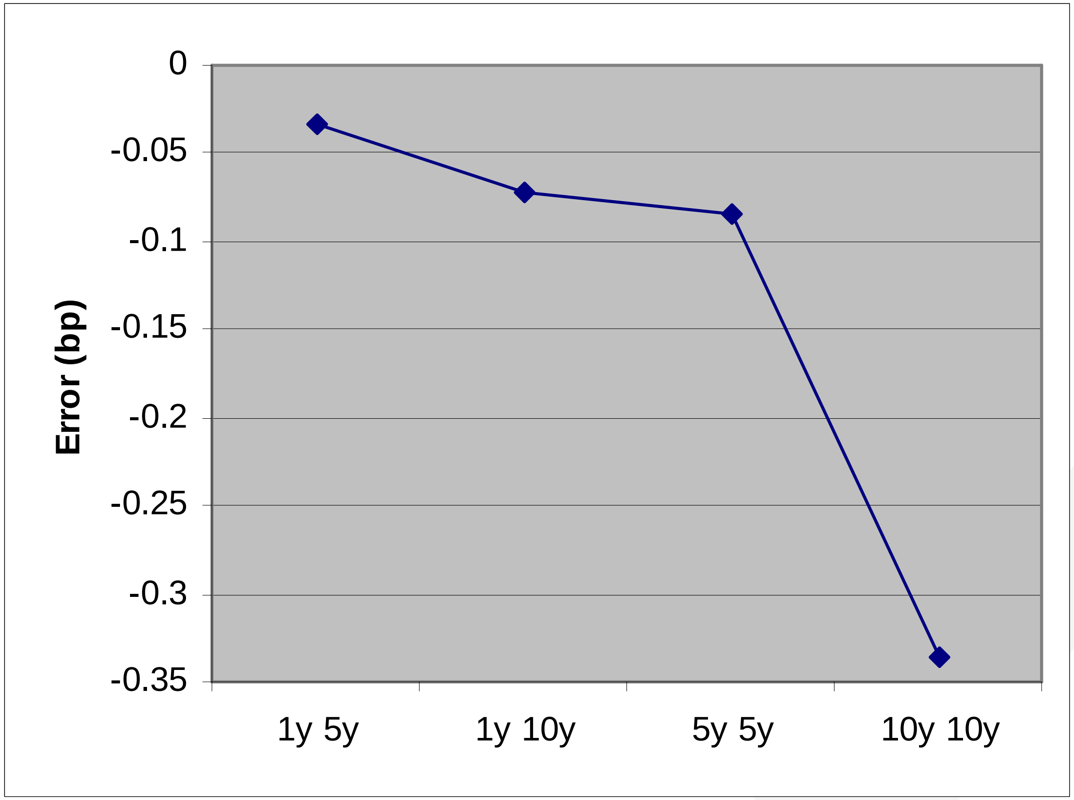
$$\rho_{ij}^{(B)} = Exp(-a\,|\,i - j\,|); \qquad a = 8 \bullet 10^{-2}$$

and the cap/floor volatilities of the 14th January 2005 at 11:15 CET.

We consider some cases:

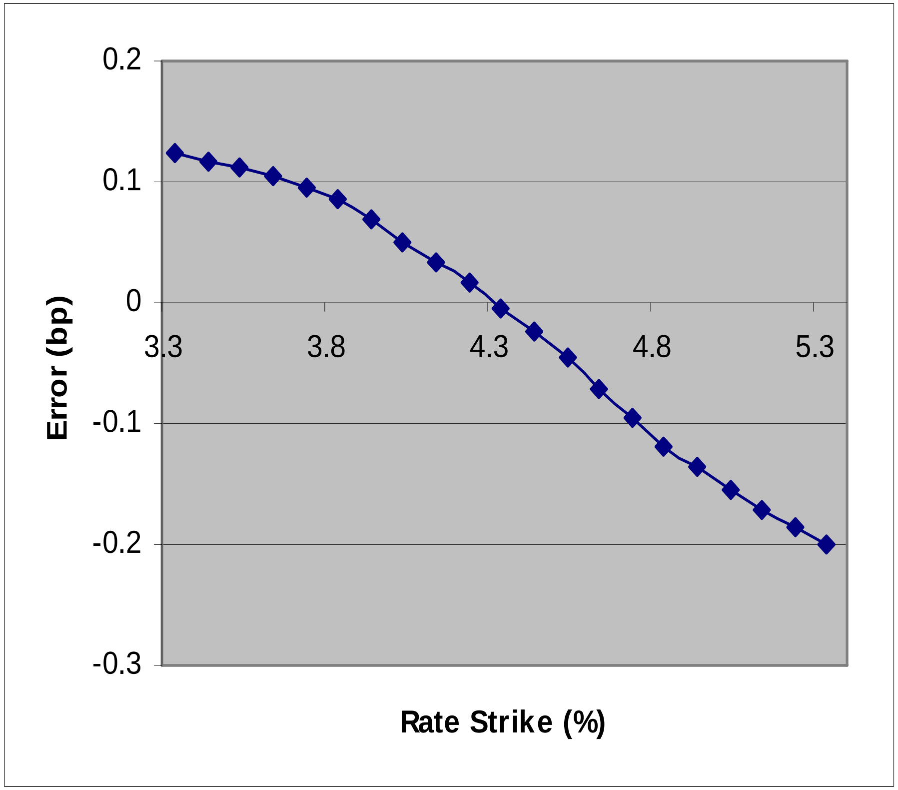
- ATM straddle
- Some very liquid swaptions *vs* Strike (5y 5y & 1y 5y)

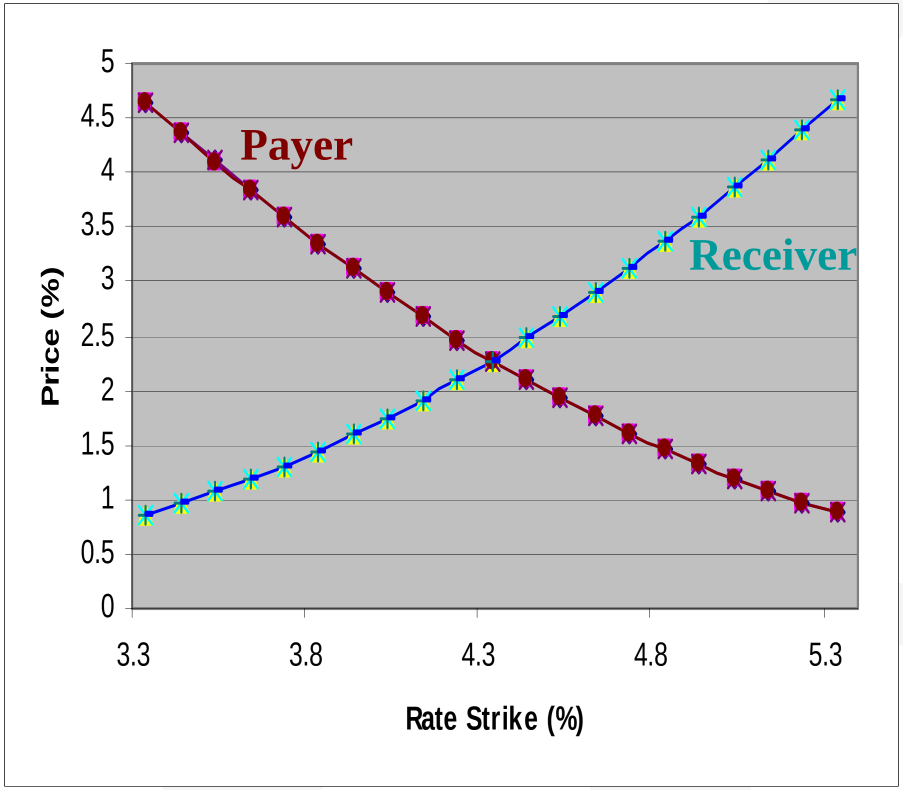
Approximation Precision: ATM Straddle



1 bp = 0.01 %

Approximation Precision: 5y 5y





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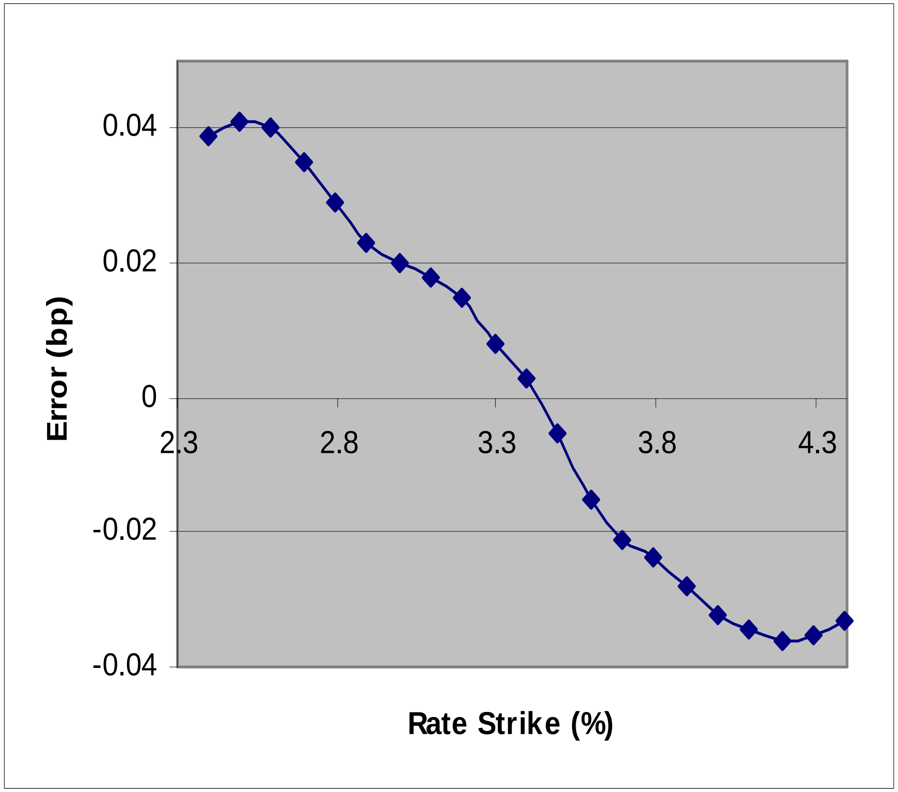
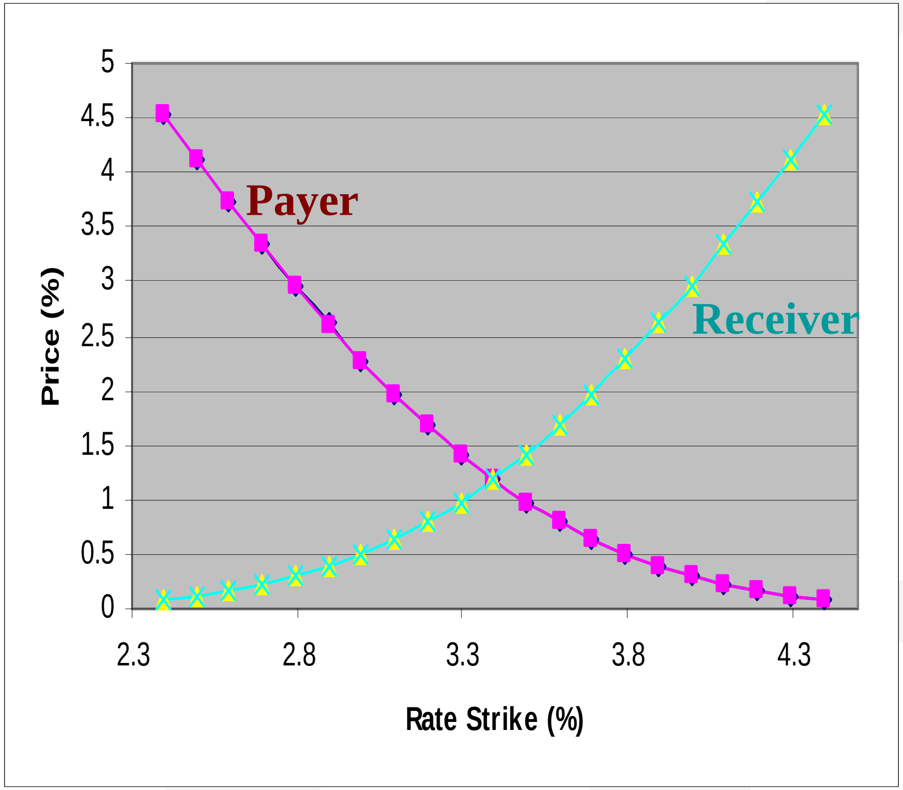
Approximation Precision: 1y 5y





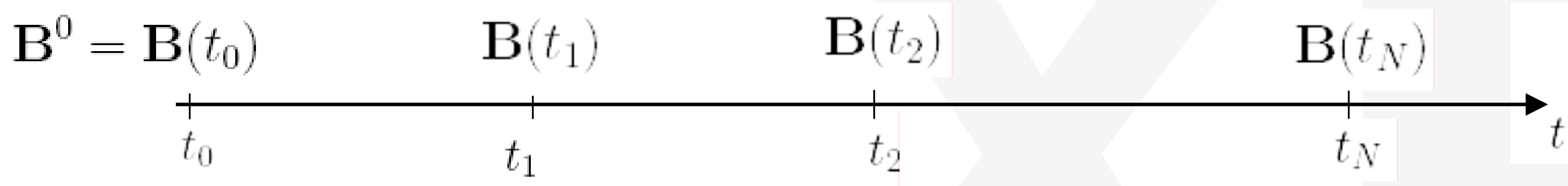
1 bp = 0.01 %

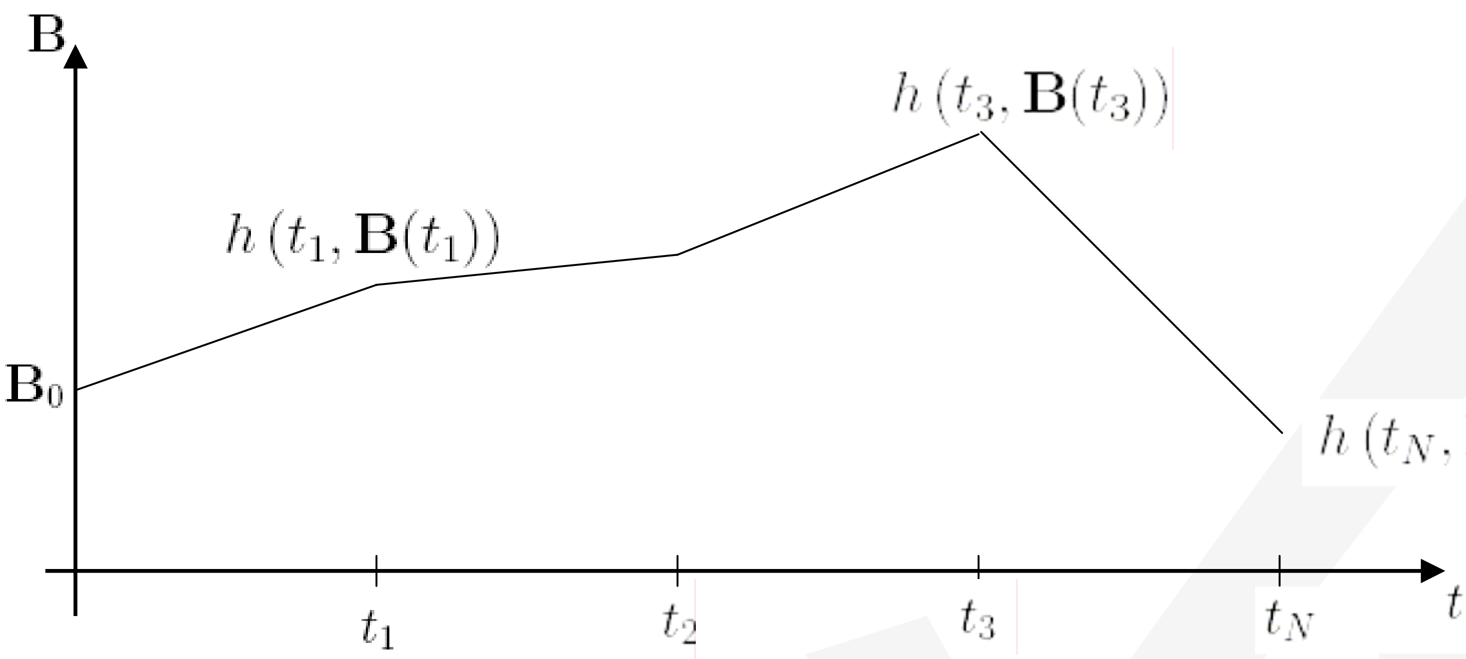
Monte Carlo: Dynamics

Reset dates: $t_0, t_1, t_2, \ldots, t_N$

Fixing

$$\mathbf{B}^0 = \begin{pmatrix} B_0(t_0) \\ B_1(t_0) \\ B_2(t_0) \\ \ldots \\ B_N(t_0) \end{pmatrix} \Rightarrow \begin{pmatrix} B_0(t_0) \\ B_1(t_1) \\ B_2(t_1) \\ \ldots \\ B_N(t_1) \end{pmatrix} \Rightarrow \begin{pmatrix} B_0(t_0) \\ B_1(t_1) \\ B_2(t_2) \\ \ldots \\ B_N(t_2) \end{pmatrix} \Rightarrow \begin{pmatrix} B_0(t_0) \\ B_1(t_1) \\ B_2(t_2) \\ \ldots \\ B_N(t_N) \end{pmatrix}$$





$$O = E\left[\sum_i D_{0\,i}\, h(t_i, \mathbf{B}(t_i))\right]$$

where $D_{0\,i}$ is the discount in (t_0, t_i)

$h(t_i, \mathbf{B}(t_i))$ is the payoff in t_i

Main Properties:

- Elementary Monte Carlo: $\forall$ fwd measure!
- Markov Chain between Reset dates $\{\mathbf{B}(t_i)\}_i$: (Lognomal) Transition Probability
- $D_{0\ i}$ function of $\{\mathbf{B}(t_i)\}_i$: $D_{0\ i} = \prod_{n=0}^{i-1} B_n(t_n)$
- Cap/Floor & Swaption: Black-like formulas
- A large set of analytical formulas Exact or Almost Exact

Callable Exotics: Problem Formulation

Bermudan option:

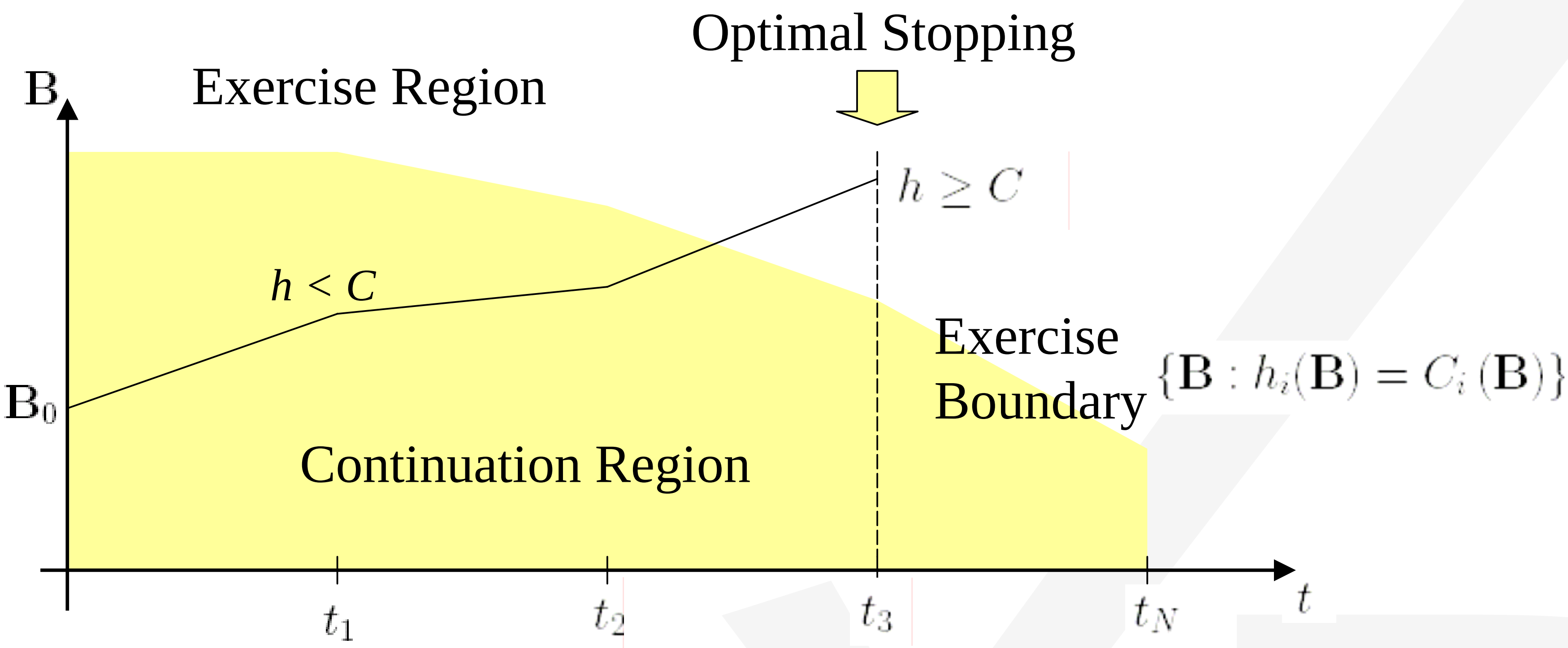
$$C_0 = \sup_{\tau \in \mathcal{T}} E_0 \left[D_{0\tau} \, h\left(\tau, \mathbf{B}(\tau)\right)\right]$$

$\mathcal{T}$: class of admissible stopping times with values in $\{t_1, t_2, \ldots, t_N\}$

Optimal stopping τ^*

$$\tau^* = \min_i \left\{ t_i : h\left(t_i, \mathbf{B}(t_i)\right) \geq C_i\left(\mathbf{B}(t_i)\right)\right\}$$

with $C_i\left(\mathbf{B}\right)$: Continuation value function



Problem:

$C_i(\mathbf{B})$ is a Bermudan option with exercise dates $\{t_{i+1}, t_{i+2}, \ldots, t_N\}$

In a MC approach each $C_i(\mathbf{B})$ should come from a **new** MC simulation starting in t_i !?!

Any approximate exercise strategy $\hat{\tau}$ provides a lower bound

$$L_0 = E_0\left[D_{0\hat{\tau}}\, h\left(\hat{\tau}, \mathbf{B}(\hat{\tau})\right)\right] \leq E_0\left[D_{0\tau^*}\, h\left(\tau^*, \mathbf{B}(\tau^*)\right)\right]$$

In the exercise decision one substitutes the continuation value function

$C_i\left(\mathbf{B}\right)$ with $\hat{C}_i\left(\mathbf{B}, \{\eta\}\right)$

Where $\{\eta\}$ are a set of parameters.

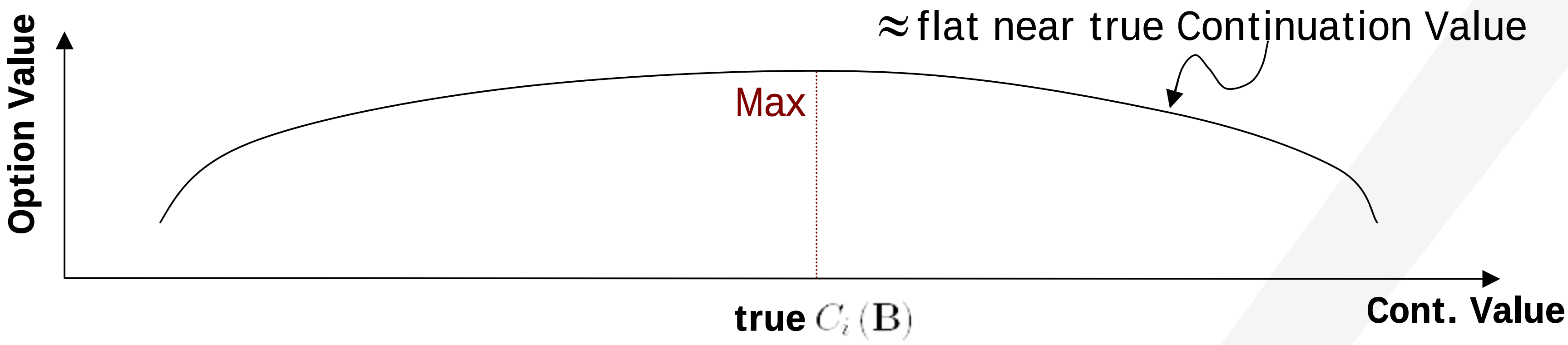
… then we find the best suboptimal exercise rule within the class. (Andersen 2000)

Idea:

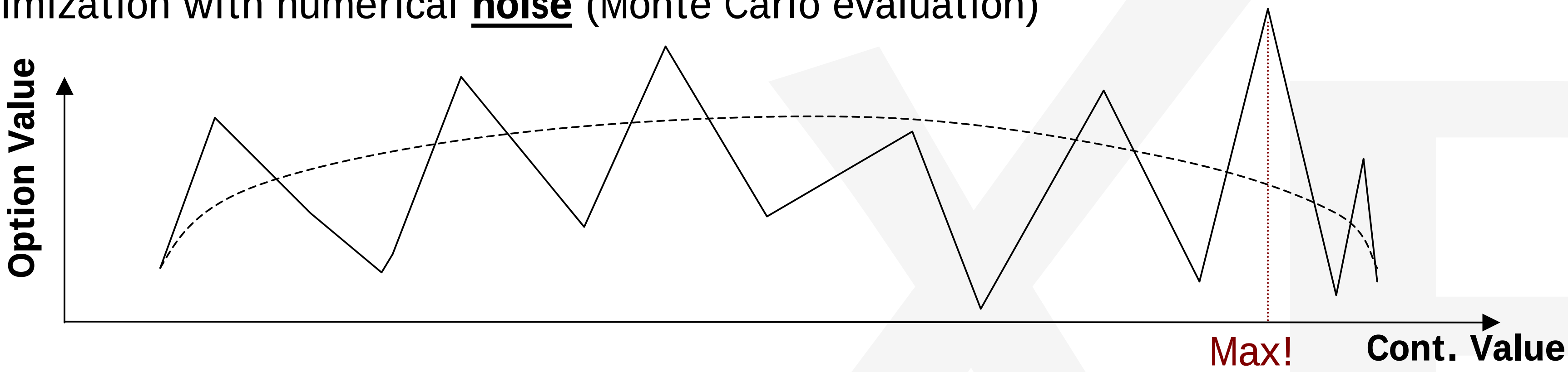
Option value <u>*not very sensitive*</u> to the **exact** position of the Exercise Boundary

Even a rough approximation of $C_i\left(\mathbf{B}\right)$ leads to a good approximation of option value

Optimization **exact** function

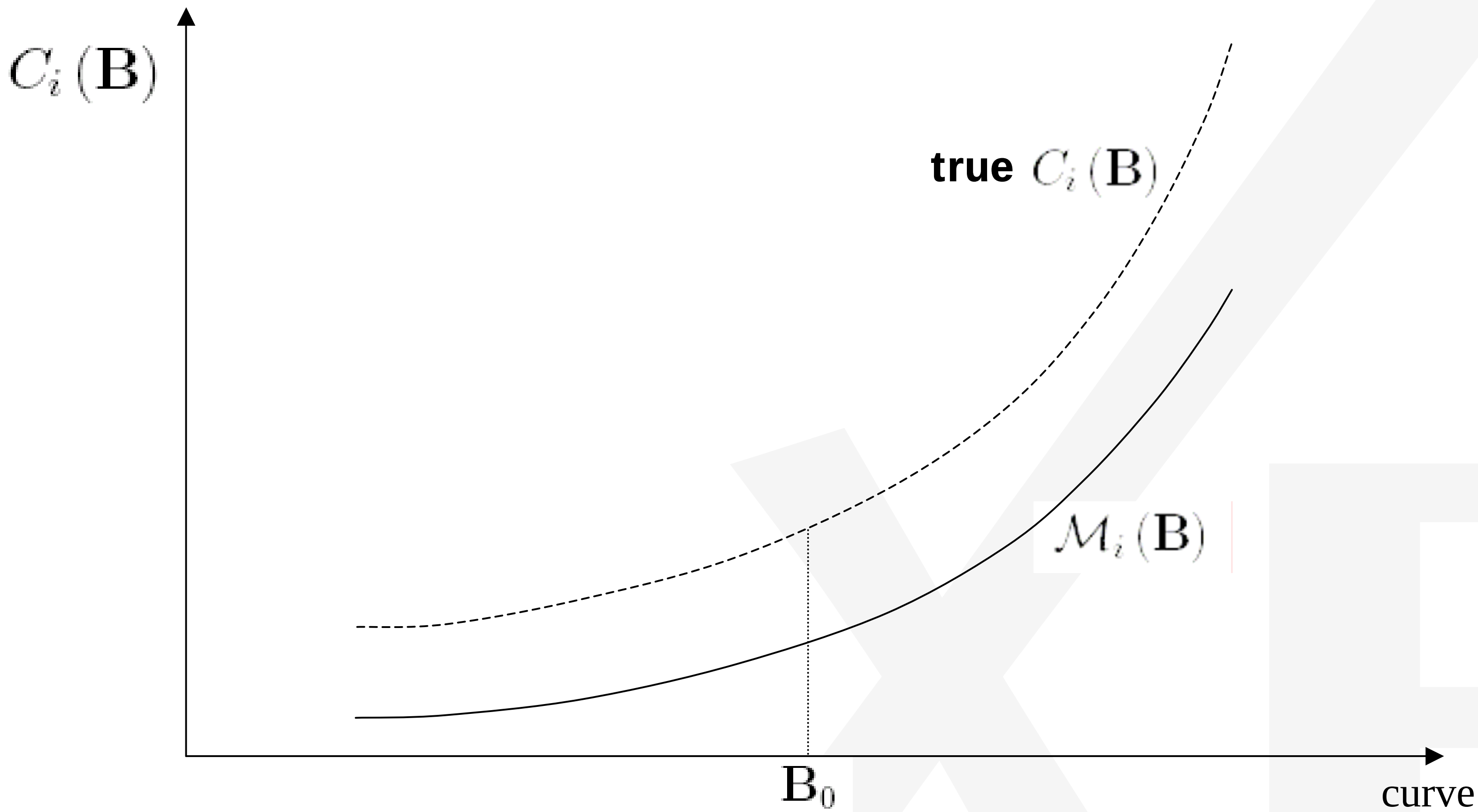


Optimization with numerical **noise** (Monte Carlo evaluation)



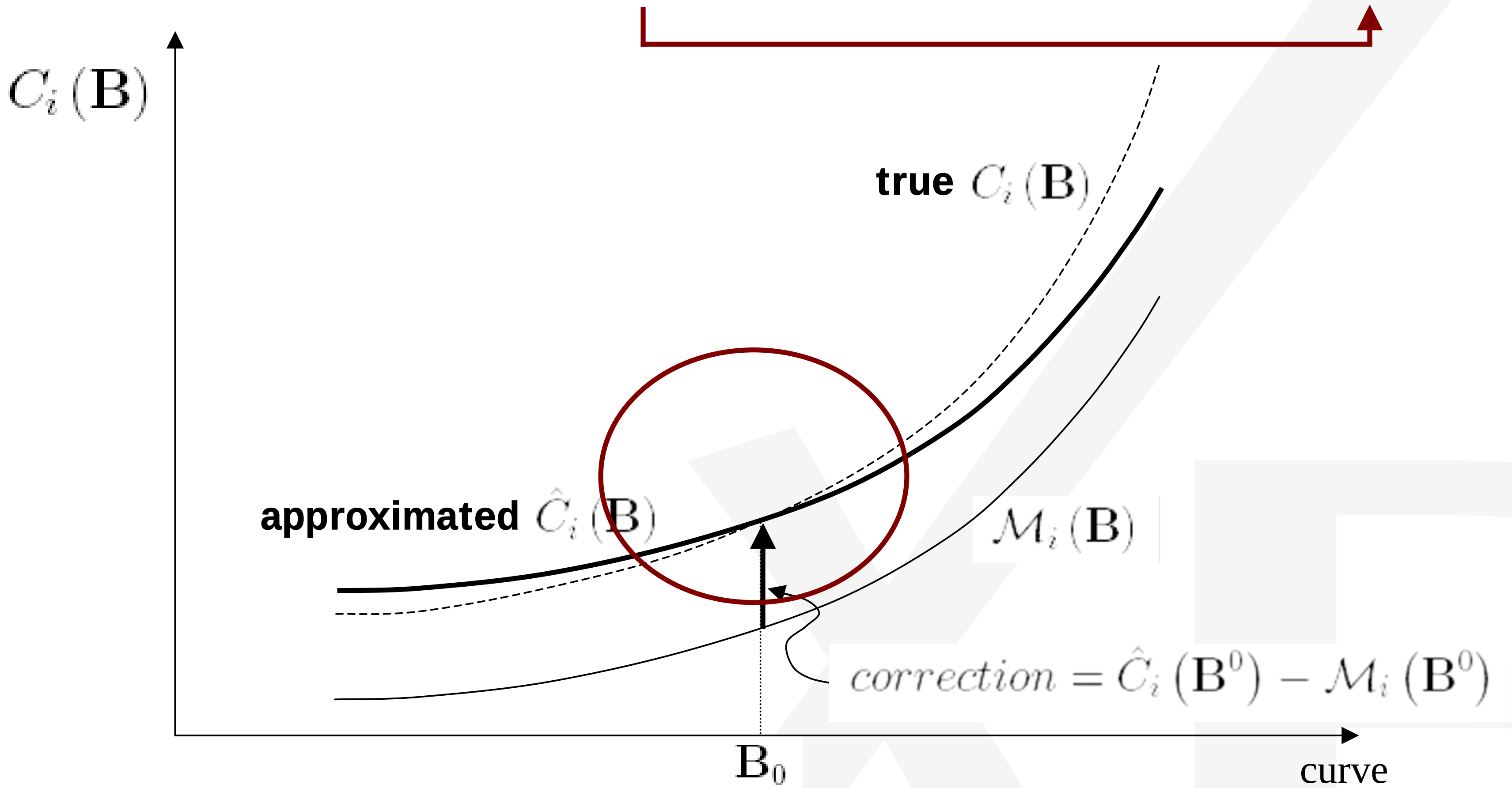
Given an arbitrary **simple** (to compute) function $\mathcal{M}_i(\mathbf{B})$

Approximated continuation value function … $\hat{C}_i^{\{0\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B})$



Given the last Continuation value function: Its a European option

Iterative backwords: $\hat{C}_n(\mathbf{B}) \quad \forall n > i$ ⇒ $\hat{C}_i(\mathbf{B}^0)$ ⇒ $\hat{C}_i(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + correction$



$\mathcal{M}_i(\mathbf{B})$ a possible choice

$$\mathcal{M}_i(\mathbf{B}) \equiv \mathcal{E}_m(\mathbf{B}, t_i) \qquad 0 < i < m \leq N$$

with $\mathcal{E}_m$ the max European option in $\mathbf{B}^0$: $\mathcal{E}_m(\mathbf{B}^0, t_i) \geq \mathcal{E}_n(\mathbf{B}^0, t_i) \;\; \forall n$

where $\mathcal{E}_n(\mathbf{B}^0, t_i)$ European option valued in t_i with expiry t_n

$$\hat{C}_i^{\{0\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B})$$

$$\hat{C}_i^{\{1\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{1\}}(i)$$

$$\hat{C}_i^{\{2\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{2\}}(i) + \sum_{n=i}^{N} c_1^{\{2\}}(i,n)\left(\ln B_n - \ln B_n^0\right) + \frac{1}{2}\sum_{n,l=i}^{N} c_2^{\{2\}}(i,n,l)\left(\ln B_n - \ln B_n^0\right)\left(\ln B_l - \ln B_l^0\right)$$

$c_0^{\{j\}}(i)$ ⇦ value in $\mathbf{B}^0$

$c_1^{\{2\}}(i,n)$ ⇦ Delta in $\mathbf{B}^0$

$c_2^{\{2\}}(i,n,l)$ ⇦ Gamma in $\mathbf{B}^0$

...

Idea:

Given Π a class of martingale processes with values in $\{t_1, t_2, ..., t_N\}$

$$\sup_{\tau \in \mathcal{T}} E_0\left[D_{0\tau}\, h\left(\tau, \mathbf{B}(\tau)\right)\right] = C_0 = \inf_{\pi \in \Pi} \left\{\pi_0 + E_0\left[\max_i \left(D_{0i} h_i - \pi_i\right)\right]\right\}$$

Lower Bound: L_0

Upper Bound: U_0

(Andersen Broadie 2004 & refs therein)

An approximated continuation value function set $\left\{\hat{C}_i(\mathbf{B})\right\}_i$

⇩

martingale process $\{\hat{\pi}_i\}_i$

$$\begin{cases} \hat{\pi}_0 = L_0 \\ \hat{\pi}_i = \hat{\pi}_{i-1} + \Delta\pi_i \quad i = 1, .., N \end{cases}$$

with:

$$\begin{aligned} \Delta\pi_i &= D_{0i+1}\left\{\max\left(h_{i+1}, \hat{C}_{i+1}\right) - E_i\left[\max\left(h_{i+1}, \hat{C}_{i+1}\right)\right]\right\} & i = 1, .., N-1 \\ \Delta\pi_N &= D_{0N}h_N - D_{0N-1}\hat{C}_{N-1} & i = N \end{aligned}$$

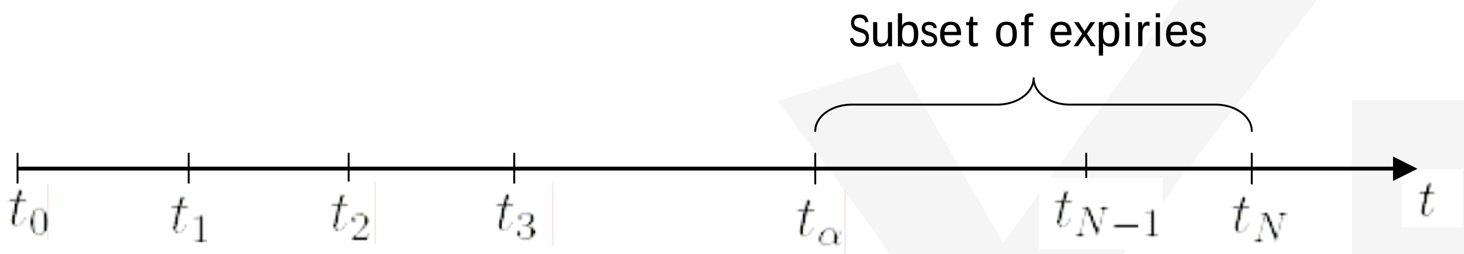
...two nested MCs

New Approach: putting numbers in

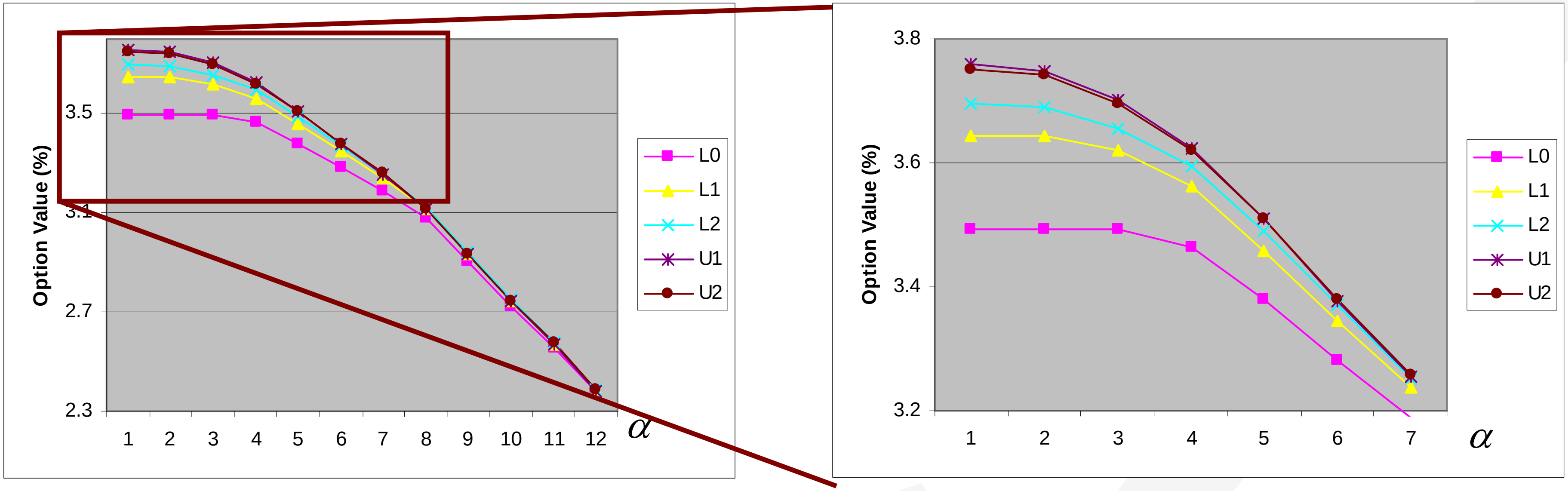
We consider some cases:

- 10y S/A Bermudan coupon option (N = 19)
- dataset of 14 Jan 05 at 11:15 CET
- and also… Bermudan options with a subset of exercise dates

$L_\alpha \, , \, U_\alpha$: first expiry in t_α



New Approach: putting numbers in



L_α using 10^6 paths

U_α using $5 \cdot 10^4$ paths (external MC) & 10^3 paths (internal MC)

Underlying: 10y S/A Step Up (3.0, 3.3, 3.7, 4.2, 4.8)

New Approach: Accuracy

Option value $$C_\alpha = L_\alpha + \frac{A}{2}$$

Accuracy in bps(*)

standard: $$A_{std} = U_\alpha - L_\alpha$$

new (estim.): $$A_{new} = L_\alpha^{\{2\}} - L_\alpha^{\{1\}}$$

α	$L_\alpha(\%)$	A_{std} (bp)	A_{new} (bp)
1	3.696	5	5
2	3.690	5	5
3	3.655	4	4
4	3.593	3	3
5	3.490	2	3
6	3.370	1	2
7	3.252	1	1

(*)1 bp = 0.01 %

Conclusions

Bond Market Model :

- A simple (multi-factor) model
- Black like formulas for caps/floors and swaptions
- MC (Markov between reset dates)
- Natural estensions - CMS related products
  - Hybrids (FX & Equity)

Callable Exotics :

- High precision
- Fast (no maximization)
- Accuracy control

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