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A simple solution for sticky cap and sticky floor

ROBERTO BAVIERA*†

Abaxbank, corso Monforte, 34, I-20122 Milan, Italy

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We show an analytical approach to *sticky cap* and *sticky floor* according to the Bond Market Model, a recently introduced version of the multi-factor Gaussian Heath–Jarrow–Morton model that is particularly easy to manage and calibrate. This solution allows having a comprehensive approach even for this class of Interest Rates' exotic derivatives that are fully path-dependent.

Keywords: Sticky cap; Sticky floor; Bond Market Model

Why is it so tough to price and manage risks involving Interest Rates' (IR) exotic derivatives? One reason is surely due to the fact that these contingent claims involve several degrees of freedom in the forward rates curve and that in the literature very few closed-form solutions are available. In this note we show how to price *sticky caps* and *floors*, IR exotic derivatives well known in the market (see e.g. Hull 1997 and Brigo and Mercurio 2001) that, due to their path-dependence, fully involve the different features of a multi-factor IR model. Furthermore, the explicit solution allows one to understand the role played by the main risk factors and in a particular way by the correlation between the different parts of the rates' term structure.

Given a set of reset dates $T_1, \ldots, T_{N+1}$, a *sticky cap* (as a plain vanilla cap) has $N+1$ possible payments, starting from K_0 paid in T_1. The nth *sticky cap* payoff K_n established at time T_n (paid in T_{n+1}) is the minimum between Libor rate with fixing at time T_n and previous payoff K_{n-1}. In a similar way for a *sticky floor*, K_n is the maximum between Libor rate with fixing at time T_n and previous rate K_{n-1}. Each payment in this exotic derivative depends then on all the previous payments: it is completely path-dependent. In this note we show that it is possible to derive an analytical solution for a *sticky cap* and a *sticky floor* in the Bond Market Model (BMM) (Baviera 2006), a multi-factor Gaussian Heath–Jarrow–Morton model (Heath *et al.* 1992) which is particularly easy to handle.

*Email: roberto.baviera@abaxbank.com
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Following the standard notation, let us define $L_n(T_n)$ as the Libor rate between T_n and T_{n+1} and fixing in T_n; for simplicity we deal with the case in which each reset date is a lag θ after the previous one.

The nth *sticky cap* payoff reads:

$$\begin{cases} \theta\, K_0, & n=0,\\ \theta\, \min(L_n(T_n), K_{n-1}), & n=1,\ldots,N, \end{cases}$$

and the nth *sticky floor* payoff is:

$$\begin{cases} \theta\, K_0, & n=0,\\ \theta\, \max(L_n(T_n), K_{n-1}), & n=1,\ldots,N, \end{cases}$$

where the nth payoff is established in T_n, calculated for the lag $\theta = T_{n+1} - T_n$ and paid in T_{n+1}. We also define $L_0(T_0) \equiv K_0$ in order to get a more compact notation.

The value of a *sticky cap* at T_0 is

$$\mathcal{C} \equiv \sum_{n=0}^{N} \theta E\left[\exp\left(-\int_{T_0}^{T_{n+1}} r_t \mathrm{d}t\right) \min_{j=0,\ldots,n} L_j(T_j)\right] \qquad (1)$$

and a *stick floor*

$$\mathcal{F} \equiv \sum_{n=0}^{N} \theta E\left[\exp\left(-\int_{T_0}^{T_{n+1}} r_t \mathrm{d}t\right) \max_{j=0,\ldots,n} L_j(T_j)\right], \qquad (2)$$

where r_t is the spot rate.

In order to evaluate equations (1) and (2) we use the BMM, a market model easy to calibrate (Baviera 2006). This model allows one to price with Black-like formulas the three classes of Over-The-Counter Interest Rates' plain vanilla options (bond options, caps/floors and

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swaptions) and analytically even some exotic payoffs (Baviera 2003).

BMM models directly the dynamics of $B_j(t)$, the forward price in t of the zero coupon bond starting in T_j which pays 1 in T_{j+1}, that is equivalent to the forward Libor rate $L_j(t)$ due to the well-known relation (see e.g. Rebonato 1996, Musiela and Rutkowski 1997 and Brigo and Mercurio 2001):

$$L_j(t) = \frac{1}{\theta}\left(\frac{1}{B_j(t)} - 1\right). \tag{3}$$

In the BMM the dynamics for $B_j(t)$, under the T_n-forward measure (Geman *et al.* 1995) is

$$\mathrm{d}B_j(t) = B_j(t)v_j(t)\left[\sum_{l=j}^{n-1}\rho v_l(t)\mathrm{d}t + \mathrm{d}W^n(t)\right] \quad \text{with} \quad n \ge j. \tag{4}$$

The volatility $v_j(t)$ is an M-dimensional deterministic function of time with $v_j(t) = 0$ for $t \ge T_j$ and W is an M-dimensional Brownian motion with instantaneous correlation $\rho = (\rho_{ij=1,\dots,M})$

$$\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{ij}\mathrm{d}t$$

and $v_j(t)\mathrm{d}W(t)$ is the scalar product between the two vectors $v_j(t)$ and $\mathrm{d}W(t)$. As shown in Baviera (2006) the volatilities $v_j(t)$ can be calibrated directly on market data, since plain vanilla caplets and floorlets are priced according to Black-like formulas. This easy calibration is mainly due to the possibility of separating volatilities $v_j(t)$ and correlations ρ_{ij}: as already stressed by Santa Clara and Sornette (2001), the ability to separately fit volatilities and correlations in IR derivatives greatly simplifies model calibration to market prices.

Let us consider an N factor model where only the jth component of the volatility $v_j(t)$ is different from zero, i.e. $(v_j(t))_l = v_j(t)\delta_{ij}$, where $v_j(t)$ is a one-dimensional function of time and δ_{ij} is the Dirichlet delta. The volatilities $v_j(t)$ can be calibrated directly on market data, using plain vanilla caplets and floorlets. The jth plain vanilla caplet/floorlet (with maturity in T_j and payment in T_{j+1}) with strike K are equal to

$$c_j = B(T_0, T_{j+1})\left[(1+\theta L_j(T_0))\mathcal{N}\left(d_1^{\mathrm{L}}\right) - (1+\theta K)\mathcal{N}\left(d_2^L\right)\right],$$

$$f_j = B(T_0, T_{j+1})\left[(1+\theta K)\mathcal{N}\left(-d_2^L\right) - (1+\theta L_j(T_0))\mathcal{N}\left(-d_1^L\right)\right],$$

where $B(T_0, T)$ is the zero coupon starting at T_0 and ending in T and

$$d_1^L = \frac{1}{\mathcal{V}_j\sqrt{T_j - T_0}}\ln\frac{1+\theta L_j(T_0)}{1+\theta K} + \frac{1}{2}\mathcal{V}_j\sqrt{T_j - T_0},$$

$$d_2^L = \frac{1}{\mathcal{V}_j\sqrt{T_j - T_0}}\ln\frac{1+\theta L_j(T_0)}{1+\theta K} - \frac{1}{2}\mathcal{V}_j\sqrt{T_j - T_0}$$

with

$$\mathcal{V}_j^2 \equiv \frac{1}{T_j - T_0}\int_{T_0}^{T_j} v_j^2(t)\mathrm{d}t.$$

The correlation ρ_{ij} can be calibrated directly on market data since

$$\rho_{ij} = \mathrm{Corr}[\mathrm{d}\ln B_i(t), \mathrm{d}\ln B_j(t)].$$

Furthermore, let us mention here that the BMM allows one to have, also under the spot measure, the conditional probability distribution of B_j (and then of the Libor rate L_j) at a reset date $T_\beta \le T_j$ given the situation at previous reset date $T_\alpha < T_\beta$: this property (which the Libor Market Model and the Swap Market Model do not have) is crucial in Monte Carlo simulations since it is possible to limit evolution only to reset dates having no discretization bias (Baviera 2006).

With empty sums denoting zero, we define the following quantities that are utilized in Proposition 1 that follows

$$\mathcal{K}_j^n \equiv \exp\left[-\int_{T_0}^{T_j} v_j(t)\rho\sum_{l=j+1}^{n} v_l(t)\mathrm{d}t\right], \quad 0 \le j \le n,$$

the volatility

$$\Sigma_i^{(n,j)\,2} \equiv \int_{T_0}^{T_n}\mathrm{Var}\left(\ln\frac{B_j(t)}{B_i(t)}\right)\mathrm{d}t = \int_{T_0}^{T_n}\left[v_j(t) - v_i(t)\right]\rho\left[v_j(t) - v_i(t)\right]\mathrm{d}t, \quad i \ne j;\ 0 \le i \le n$$

and the correlation matrix R

$$\mathsf{R}_{il}^{(n,j)} \equiv \frac{1}{T_n - T_0}\int_{T_0}^{T_n}\mathrm{Corr}\left(\ln\frac{B_j(t)}{B_i(t)}, \ln\frac{B_j(t)}{B_l(t)}\right)\mathrm{d}t = \frac{1}{\Sigma_i^{(n,j)}\Sigma_l^{(n,j)}}\int_{T_0}^{T_n}\left[v_j(t) - v_i(t)\right]\rho\left[v_j(t) - v_l(t)\right]\mathrm{d}t,$$
$$i, l \ne j;\ 0 \le i, l \le n.$$

Proposition 1: *In the BMM the sticky cap is equal to*

$$\mathcal{C} \equiv \sum_{n=0}^{N} B(T_0, T_{n+1})\left\{\sum_{j=0}^{n}\left[1+\theta L_j(T_0)\right]\mathcal{K}_j^n\mathcal{N}_n(\mathbf{d}^{(n,j)}, \mathsf{R}^{(n,j)}) - 1\right\} \tag{5}$$

and the sticky floor is

$$\mathcal{F} \equiv \sum_{n=0}^{N} B(T_0, T_{n+1})\left\{\sum_{j=0}^{n}\left[1+\theta L_j(T_0)\right]\mathcal{K}_j^n\mathcal{N}_n(-\mathbf{d}^{(n,j)}, \mathsf{R}^{(n,j)}) - 1\right\} \tag{6}$$

where $\mathcal{N}_n(\mathbf{d}, \mathsf{R})$ stands for the n-variate normal distribution (with $\mathcal{N}_0(\cdot) \equiv 1$) evaluated with correlation matrix $\mathbf{R}$ and at the vector $\mathbf{d}$ where

$$d_i^{(n,j)} \equiv \frac{1}{\Sigma_i^{(n,j)}}\ln\left(\frac{B_j(T_0)\mathcal{K}_i^n}{B_i(T_0)\mathcal{K}_j^n}\right) - \frac{1}{2}\Sigma_i^{(n,j)}, \quad i \ne j,\ 0 \le i \le n.$$

Proof: Let us consider the cap case since *mutatis mutandis* the proof is the same for the floor. Recalling relation (3) between Libor rates and forward zero coupons, equation (1) is equivalent to

$$\mathcal{C}=\sum_{n=0}^{N}\left\{E\left[\exp\left(-\int_{T_0}^{T_{n+1}} r_t \mathrm{d}t\right)\min_{j=0,\dots,n}\frac{1}{B_j(T_j)}\right]-B(T_0,T_{n+1})\right\}.$$

The above equation, applying the change of numeraire technique (Geman *et al.* 1995), can be written as

$$\mathcal{C}=\sum_{n=0}^{N}B(T_0,T_{n+1})\left\{E^{(n+1)}\left[\min_{j=0,,n}\frac{1}{B_j(T_j)}\right]-1\right\},$$

where $E^{(n+1)}[\cdot]$ is the expectation under the T_{n+1}-forward measure (Geman *et al.* 1995). Let us consider the case where the lowest value is the *j*th element. In this case the minimum condition reads

$$B_j(T_j)>B_i(T_i),\quad \forall i\neq j$$

or equivalently, given the dynamics (4) for B_j,

$$-\Sigma_i^{(n,j)}z_i\equiv-\int_{T_0}^{T_n}\left[v_i(t)-v_j(t)\right]\mathrm{d}W^{(n+1)}(t)$$
$$<\ln\frac{B_j(T_0)\mathcal{K}_i^n}{B_i(T_0)\mathcal{K}_j^n}+\frac{1}{2}\int_{T_0}^{T_n}\left[v_j(t)\rho v_j(t)-v_i(t)\rho v_i(t)\right]\mathrm{d}t$$

with z_i unitary variance Gaussian variables with correlation $\mathrm{Corr}(z_i,z_l)=\mathsf{R}_{il}^{(n,j)}$.

The proposition is proven applying the Girsanov transform

$$\mathrm{d}\tilde{W}^{(n+1)}(t)=\mathrm{d}W^{(n+1)}(t)+\rho v_j(t)\mathrm{d}t. \qquad \square$$

We have then shown that *sticky* derivatives are similar to equity options best/worst of several assets: as shown in equations (5) and (6), the correlation ρ in *stickies* plays a crucial role like in the equity case. *Stickies* have been the first (and simplest) correlation product in the IR derivatives market: option's writer is short correlation in a *sticky cap* and long correlation in the *sticky floor* case.

Unfortunately, it is quite hard hedging the correlation exposure resulting from these derivatives through plain vanilla swaptions: as a matter of market curiosity, it can be interesting to notice that in these last few years, after several issues of structured bonds with *sticky caps*, it has become quite common to find payoffs like *snowball* where option's writer is long correlation; these payoffs result to be a natural hedge for *sticky cap* correlation exposure.

In this note we have derived an analytical solution for both *sticky cap* and *sticky floor* in the Bond Market Model. The analytical approach proposed offers closed form solutions for IR contingent claims (even when path dependence features are involved), it allows one to decompose the different sources of risk and it clarifies how to manage them.

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