Option Prices in Presence of Transaction Costs

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Abstract

We provide closed formulas for European call option ask and bid prices in presence of transaction costs.

Underlying prices have the same dynamics of Black-Scholes model and a bid-ask spread proportional to bid price. We suppose that a market maker has to quote a bid and ask price for an option in a perfect competition market. Under these conditions derivative prices are obtained imposing the No Almost Sure Arbitrage Principle: the market maker fixes bid (ask) price as the highest buying (lowest selling) price that can be accepted by an investor who maximizes the growth rate of his portfolio.

1 Introduction

Since Black-Scholes option pricing [4], an open problem has been how to generalize their results to an underlying with transaction costs. In this paper we provide closed formulas for bid and ask option prices in this more general case.

We consider a market with two securities: a riskless bank account paying constant interest and a risky asset (e.g. shares of a corporation) with a price described by a Geometric Brownian Motion and a bid-ask spread proportional to bid price. We assume a null interest rate for notational simplicity.

In this two-assets market and in presence of perfect competition, a market maker has to quote bid and ask prices for a derivative contract written on the risky asset as underlying. In this paper we focus on an European call option with physical delivery at the maturity and we deduce in this case closed formulas for bid and ask prices (see (25 and eq. (26)).

This problem is not new in the literature and it was already clear that the Black and Scholes methodology to price derivatives cannot be used in presence of transaction costs since it is not possible anymore a continuous-time perfect replication of the derivative with stocks and bank-account. Leland [12] has first proposed to limit the replication only to a discrete sequence of times separated by a fixed lag. This method is far from optimal: after a fixed lag underlying prices can have not changed significantly (and then trade is useless) or have changed too much (and then it is probably too late to hedge!).

A more sophisticated approach has been introduced by Davis et Al. [9], who optimize market maker's portfolio. This portfolio includes the bank account, the underlying and the obligation to deliver one share for the strike price K if share price is greater than K at the maturity. Davis et Al. outline that, since perfect hedging is no longer possible in the presence of transaction costs, the arbitrage argument alone is not enough to uniquely determine derivative's price and then option pricing involve a preference-dependent valuation and "investor's attitude toward risk must be considered" [9]. In particular, in their model, the market maker has an exponential utility.

This approach has two main limits. The first limit is related to the way costs are introduced in the model. At each time the problem is described by a continuous time Bellman-Hamilton-Jacobi equation for the exponential utility function, where a dissipation term takes into account the costs payed up to that moment by the market maker in his portfolio optimization. This introduces a non-trivial memory since one has to remember the costs of the strategy since the beginning of the contract: in this way they are not able to write close pricing formulas. The second limit is that Davis et Al. do not allow derivative trading during the life of the contract: the market maker sells one option at the beginning of the contract and keeps this position until the maturity; any market participant should optimize his portfolio taking into account this freedom, treating the option as any other asset.

We overcome these limits facing market maker's problem from a different perspective. First, instead of a partial-differential-equation approach as in Davis et Al., we follow a probabilistic approach: in this way we can show that bid and ask prices can be written as expected values of the final payoff even in presence of transaction costs. Second, we reverse the point of view in the pricing problem from the market maker to the other market participants. Since in presence of transaction costs a market maker can no longer replicate the derivative, prices will be the ones for which the "market" can agree to trade: the bid will be the highest buying price and the ask the lowest selling price that can be accepted by an investor.

In particular, in this paper we consider investors who, having an infinite horizon, try to maximize the growth rate of their capital. We call them speculators.

This new approach has been first introduced in [1] in the incomplete market framework; derivative prices are fixed by the Principle of *No Almost Sure Arbitrage* (hereafter NASA): *prices of derivative securities must be such that a speculator cannot construct a portfolio out of combinations of the derivatives and the underlying security which grows almost surely at a faster exponential rate than a portfolio containing only the underlying security.* Moreover, in a market where perfect competition among market makers takes place, ask price will be the lowest and bid price the highest among all possible ones.

In presence of transaction costs, the pricing process is a game with speculators on one side and market makers in perfect competition on the other. Let us show how this game determines derivative prices (e.g the ask price) in the case of the simplest contingent claim: a forward contract.

How a market maker should fix forward price given ask and bid stock prices? We notice that in the market described above, shares and forwards are equivalent for a speculator. Forward ask price F^a will be never lower than stock bid price S^b, otherwise arbitrage would be possible. If F^a is lower than ask stock price S^a (but higher than bid stock price), this forward ask price does not imply a possibility of arbitrage for a speculator, since he cannot make money for sure buying forwards and selling stocks. However, whenever the speculator should buy shares, it is more convenient for him to buy an equivalent amount of forwards; since all speculators will buy forwards instead of shares, forward price will rise. Thus forward ask price can be only greater or equal than share ask price, but in presence of perfect competition among market makers, it will be the lowest possible, i.e. stock ask price.

We have shown that forward ask price is equal to the underlying price at each time (e.g. at the beginning of the contract, i.e. time 0), or equivalently the expected value of the payoff at the maturity T under a martingale measure

$$F^a = S_0^a = E_{Q_a}[\tilde{S}_T^a]$$

where $\tilde{S^a}$ is a martingale process equivalent to S^a, martingale w.r.t. Q^a. In this paper, we generalize the same result to a call option (and then to a contingent claim with a generic payoff at the maturity), showing that

$$\begin{array}{c} C_0^a = E_{\mathcal{Q}_a}[(\tilde{S}_T^a - K)^+] \\ C_0^b = E_{\mathcal{Q}_b}[(\tilde{S}_T^b - K)^+] \end{array} \tag{1}$$

with

$$\begin{array}{c} S_0^a = E_{\mathcal{Q}_a}[\tilde{S}_T^a] \\ S_0^b = E_{\mathcal{Q}_b}[\tilde{S}_T^b] \end{array} .$$

and we also specify the measures $\mathcal{Q}_a$ and $\mathcal{Q}_b$.

The rest of the paper is organized as follows: in section **2** we recall the results of portfolio optimization in the case where shares and risk-free assets are the only securities that can be traded. This is a crucial step since we want to compare the growth of a portfolio without derivative with one where it is included. In section **3** we introduce some new derivative contracts; we call them *synthetic* call options. These derivatives are simple to price according to NASA Principle and appear to be the fundamental brick in our derivation of option prices in presence of transaction costs. This derivation is presented in section **4**. Finally, in section **5** we state some concluding remarks.

2 Portfolio without derivative

In this section we recall some results on portfolio selection in presence of transaction costs in a market with two assets: a bank account and a risky asset.

Ask and bid price are proportional

$$S^a = e^\gamma S^b \tag{2}$$

where γ is related to risky-asset bid-ask spread and bid stock price follows a Geometric Brownian motion

$$dS_t^b = (\mu + \sigma^2/2) S_t^b dt + \sigma S_t^b dw_t \ , \tag{3}$$

where μ is the drift of the log-price, σ is the standard deviation and w_t is a Wiener process. We do not allow the speculator to borrow money from a bank or short selling of stock. This is equivalent [5] to considering

$$-\frac{\sigma^2}{2} \leq \mu \leq \frac{\sigma^2}{2} \, . \tag{4}$$

A larger drift implies that the best solution is to transfer all the money to the stock. Instead the opposite limit corresponds to having no money in the risky asset.

In this paper we select the optimal trading strategy following the repeated game approach introduced by Kelly [7]. Following Kelly [7] we consider a speculator who desires to maximize the growth rate of his capital

$$\lambda \equiv \lim_{\Omega \to \infty} \frac{1}{\Omega} \ln \frac{W_{(\Omega)}}{W_{(0)}} \tag{5}$$

where $W_{(t)}$ is the capital at time t.

Kelly's approach is the most natural if an investor has an infinite horizon and at each trade repeats exactly the same "game": his portfolio diversification in the two assets. Under these conditions, in the long run the capital will *almost surely* grow exponentially with a rate λ, since the variance of the growth rate goes to zero with the number of trades. The portfolio selection problem is then equivalent to selecting the fraction l of capital invested in the stock (the remaining capital is left in the bank account) and to choosing the optimal timing to trade again, since in presence of transaction costs, it is (obviously) better not to trade continuously.

This problem, first addressed by [15], who have written the Hamilton-Jacobi-Bellman equation for capital's growth rate, has been faced in a more general framework by [8, 11, 5]. However, following this approach, no fully analytic solution has been found even in the simplest cases.

Recently we have proposed an approach [2], different from the standard one, which allows the problem to be completely solved by mapping it into one without costs. In presence of a bid-ask spread, stock price is not unique; however, in the optimization of speculator's portfolio only prices of traded assets are relevant and, in this case, asset *value* S is uniquely defined. In the other cases, asset *value* is a matter of convention. Following [2] the convention we adopt is:

- at a time when a trade occurs, assets in speculator's portfolio have the same *value* of the ones just traded (marked to market)
- between two trades, asset *value* is ask price if last trade is an ask, bid price if otherwise.

As already shown in the literature [15, 8, 11, 5], in the optimal trading the speculator has to modify his portfolio only when prices change of a "relevant" amount (reaches a barrier), in particular he has to sell after a price increases (upper barrier) and buy otherwise (lower barrier). The time between two transactions is a stochastic process; it corresponds in the mathematical literature to the exit time of a 1-D diffusion process with two barriers [6].
Let us recall the main results in this portfolio selection following a two-step optimization as in [2] : we first select the optimal fraction to be invested in the risky security for (generic) finite barriers and then choose their optimal values (i.e. the timing of the trading rule).
Let us define trading *state* (or briefly *state*) the side (bid/ask) in a transaction. As shown in [2], the optimal trading rule for the speculator depends on the *state* of the current transaction (e.g. the m^{th}). If the *state* is an ask, before trading again the speculator has to wait a log-price variation $\Delta+\gamma$ if it is positive or $-\delta$ if negative. In the same way, if the transaction is a bid, the two barriers for log-prices are δ if price increases and $-(\Delta+\gamma)$ otherwise.
Thus, between two successive trades (e.g. the m^{th} and the $(m+1)^{th}$), speculator's capital can evolve in four different ways depending on the *state* (ask/bid) of the current trade (ϕ) and of the next trade (ψ). The problem is then completely defined by a Markovian transition matrix between the current and next *state*

$$V = \begin{pmatrix} 1-\pi_a & \pi_a \\ \pi_b & 1-\pi_b \end{pmatrix} , \qquad (6)$$

where π_a and π_b are the transition probabilities defined in appendix **A**.

The probabilities of the *states* a and b satisfy the following relation :

$$\frac{\nu_a}{\nu_b} = \frac{V_{ba}}{V_{ab}} = \frac{\pi_b}{\pi_a} \ .$$

Defining capital's return

$$\omega \equiv \frac{W_{m+1}}{W_m} \ ,$$

where W_m is the capital at the m^{th} trade, then the capital growth rate for a speculator (defined in 5) who follows this strategy is:

$$\lambda = \frac{1}{\mathcal{T}} \sum_{\phi,\psi} \nu_\phi V_{\phi\psi} \ln \omega_{\phi\psi} \ , \tag{7}$$

where the average exit time $\mathcal{T}$ is

$$\mathcal{T} = \sum_{\phi} \nu_\phi \mathcal{T}_\phi$$

with $\mathcal{T}_a$ ($\mathcal{T}_b$) being the average time before the next transaction occurs when the current trade is an ask (bid) (see Table **A.1** in appendix **A**).

The four values that ω can assume depending on the current (ϕ) and the next *state* (ψ) are

ϕ	ψ	$\omega_{\phi\psi}$	$u_{\phi\psi}$
ask	*ask*	$\omega_{aa} \equiv 1 + l_a(u_{aa} - 1)$	$u_{aa} \equiv e^{-\delta}$
ask	*bid*	$\omega_{ab} \equiv 1 + l_a(u_{ab} - 1)$	$u_{ab} \equiv e^{\Delta}$
bid	*ask*	$\omega_{ba} \equiv 1 + l_b(u_{ba} - 1)$	$u_{ba} \equiv e^{-\Delta}$
bid	*bid*	$\omega_{bb} \equiv 1 + l_b(u_{bb} - 1)$	$u_{bb} \equiv e^{\delta}$

where l_a (l_b) is capital's fraction invested in stocks if the current trade is an ask (bid) and $u_{\phi\psi}$ is the ratio between the next risky-asset *value* and the current one in the four cases, i.e. stocks' return after having taken into account transaction costs.

Growth rate (7) is optimal w.r.t. $\{l\}$ for

$$0 = \frac{\partial \lambda}{\partial l_{\phi'}}|_{\{l_\phi = l^*_\phi\}_{\phi=a,b}} = E_V[\frac{u_{\phi'} - 1}{1 + l^*_{\phi'}(u_{\phi'} - 1)}] \ \text{for} \ \phi' = a, b \ , \tag{8}$$

where $u_{\phi'}$ is the return (as random variable) starting from the state ϕ'. The optimal growth rate becomes

$$\lambda^* = \frac{1}{\mathcal{T}} \sum_{\phi,\psi} \nu_\phi V_{\phi\psi} \ln \frac{V_{\phi\psi}}{Q_{\phi\psi}} \ , \tag{9}$$

where we have defined the transition probability matrix

$$Q \equiv \begin{pmatrix} \dfrac{1-\pi_a}{1+l_a^*(u_{aa}-1)} & \dfrac{\pi_a}{1+l_a^*(u_{ab}-1)} \\ \dfrac{\pi_b}{1+l_b^*(u_{ba}-1)} & \dfrac{1-\pi_b}{1+l_b^*(u_{bb}-1)} \end{pmatrix} = \begin{pmatrix} 1-q_a & q_a \\ q_b & 1-q_b \end{pmatrix} , \tag{10}$$

with

$$\begin{aligned} q_a &\equiv \frac{e^{\delta}-1}{e^{\delta+\Delta}-1} \\ q_b &\equiv \frac{1-e^{-\delta}}{1-e^{-\delta-\Delta}} \end{aligned} . \tag{11}$$

The last equality in eq. (10) is obtained substituting the optimal fractions $\{l^*\}$ (see Table **A.1** in appendix **A**). From eq. (8) it is straightforward to show that

$$\begin{aligned} E_Q[u|a] = Q_{aa}u_{aa} + Q_{ab}u_{ab} = 1 \\ E_Q[u|b] = Q_{ba}u_{ba} + Q_{bb}u_{bb} = 1 \end{aligned} .$$

In this sense, u_a (u_b respectively) is a martingale process w.r.t. Q given a (given b).

Finally, the optimal solution is obtained by choosing the optimal barriers. The optimal growth rate is attained for $\delta = 0$ and Δ satisfying the equation

$$\frac{2\mu}{\sigma^2} \sinh\left(\frac{\Delta}{2}\right) = \sinh \frac{\mu}{\sigma^2}(\Delta+\gamma) \quad , \tag{12}$$

whose explicit form for small γ is

$$\Delta = \left[\frac{24\gamma}{1-(\frac{2\mu}{\sigma})^2}\right]^{\frac{1}{3}} .$$

i.e. Δ goes to zero as $\gamma^{1/3}$ when γ goes to zero.

In this way we have been able to answer to the two questions of portfolio management without derivative in presence of transaction costs since we have obtained the optimal diversification (see Table A.1) and the optimal timing (see eq. (12)). Finally the problem is completely solved writing capital's growth rate in the optimal strategy (see eq. (28)). In appendix **A** we summarize the results.

Let us only stress one central result: in the limit where δ is sent to zero, the average time between two transactions goes to zero and then the speculator trades an infinite number of times in every fixed lag Ω; for this reason *a posteriori* Kelly's approach is well justified even in the finite horizon case.

The remainder of the paper follows a similar derivation in the case of a portfolio including derivatives: first we consider δ finite and we show the optimal portfolio selection; only at the end we choose the optimal barriers' value and then the optimal timing of the strategy. In the next section we solve the pricing problem for some ad hoc defined *synthetic options* and we show in section **4** how to derive option price.

3 Synthetic options

As already stressed in the introduction, option pricing in presence of transaction costs is the result of a game between a speculator and a market maker.

On one side, a market maker fixes option prices in such a way that a speculator cannot take advantage of bid prices that are too high and ask prices that are too low; however, he cannot choose a spread too large since, in a market with perfect competition, other market makers can undercut his spread. On the other side, a speculator selects an optimal portfolio among underlying, derivative and riskless asset; a diversification which depends on the prices fixed by the market maker.

As already underlined in the introduction, in this paper the market maker chooses ask and bid prices according to the NASA Principle. In this section we show how this principle works in the case of some "ad hoc"-defined *synthetic options*.

Synthetic options are introduced only when the speculator, who manages a portfolio with only underlying and risk-free asset, trades on the underlying. As described in the previous section, he limits his trades to the times when one of the two barriers is reached; also in this section we first consider finite barriers taking their optimal values only at the end.

Let us focus on his m^{th} trade, assuming that the speculator buys the underlying. In this case he can also buy a *synthetic call option* $\mathcal{A}^{(1)}$ which expires at his next trade (i.e. when log-price varies of $\Delta+\gamma$ or $-\delta$) and gives the right to buy the underlying at strike price K, if log-price reaches the upper $(\Delta+\gamma)$ or lower $(-\delta)$ barrier.

A *synthetic options* has a maturity T which depends on price variations and then T is not *a priori* fixed as in a standard plain vanilla call option.

However the payoff is the one of a call with physical delivery:

- if log-price reaches the upper barrier the payoff is $(S^b_T - K)^+$, since the speculator will sell the underlying at the next trade and then underlying *value* is equal to the bid price (even in option's payoff!);

- if log-price reaches the lower barrier the payoff is $(S^a_T - K)^+$ since the speculator will buy the underlying in this case.

In table **1** we report in detail the payoffs of this option given the fact that the speculator is buying the underlying at his m^{th} trade (the current state $\phi = ask$) and depending his next trade (the next state ψ).

ϕ	*Price Variation*	ψ	*Payoff at the maturity*
ask	$+$	bid	$\mathcal{A}^{(1)}_T \equiv (S^b_T - K)^+ = (S^b_0 e^{\Delta+\gamma} - K)^+ = (S^a_0 e^{\Delta} - K)^+$
ask	$-$	ask	$\mathcal{A}^{(1)}_T \equiv (S^a_T - K)^+ = (S^a_0 e^{-\delta} - K)^+$

Table **1**: Payoff of *synthetic options* $\mathcal{A}^{(1)}$. We report the current (ϕ) and next *state* (ψ) of the transaction on the underlying and the sign of price variation. The last equality in the first row is obtained using eq. (2).

The speculator will play a repeated game even when this derivative is present. To keep the notation simpler, we assume that, every time the transaction on the underlying is an ask, the speculator can include in his portfolio a *synthetic* option with the same moneyness (S_0/K). Thus the speculator, besides a fraction l of his capital invested on the underlying, has the possibility to buy *synthetic* options $\mathcal{A}^{(1)}$ with a fraction $d > 0$; the remaining part $1 - l - d$ is kept in the bank account. Since option's return

$$f \equiv \frac{\mathcal{A}^{(1)}_T}{\mathcal{A}^{(1)}_0} \tag{13}$$

assumes the same two values at each trade in this repeated game, the fraction d invested in this security will be the same at each time. In a way similar to the previous section, even with the derivative in speculator's portfolio, the capital variation $\omega^{\mathcal{D}}$ can assume four different values depending on the current and next trade's state: if the current *state* is an ask we obtain

ϕ	ψ	$\omega^{\mathcal{D}}_{\phi\psi}$
a	a	$\omega^{\mathcal{D}}_{aa} \equiv 1 + l_a(u_{aa} - 1) + d(f(u_{aa}) - 1)$
a	b	$\omega^{\mathcal{D}}_{ab} \equiv 1 + l_a(u_{ab} - 1) + d(f(u_{ab}) - 1)$

and $\omega^{\mathcal{D}} = \omega$ otherwise; capital's growth rate is

$$\lambda^{\mathcal{D}}(\{l_\phi\}_{\phi=a,b}, d; f) = \frac{1}{\mathcal{T}} \sum_{\phi,\psi} \nu_\phi V_{\phi\psi} \ln \omega^{\mathcal{D}}_{\phi\psi} \ . \tag{14}$$

Let us apply the NASA Principle in a perfect competition market: the price (and then the return f as a function of the random variable u_ϕ) fixed by the market maker must be such that the maximum growth rate including the *synthetic* option (14) equals λ^*, the optimal rate in absence of derivative (see eq. (9)). Since one has $\lambda^{\mathcal{D}}(\{l_\phi = l^*_\phi\}_{\phi=a,b}, d = 0^+; f) = \lambda^*$, the maximum of (14) must be in $\{l_\phi = l^*_\phi\}_{\phi=a,b}$ and $d^* = 0^+$.

The fact that in the optimal strategy the speculator should buy almost zero options should not surprise. From a technical point of view, setting d^* to zero, is closely related to Samuelson's methodology [13] of warrant pricing in the 'incipient case'. This idea has been reconsidered more recently by Davis [10] in the incomplete markets option pricing framework. In our case the condition $d^* = 0$ stems from this two players game where the market maker fixes derivative prices in such a way that speculator cannot increase his capital growth rate investing in the option.

The condition for l^*_a is

$$0 \ = \ \frac{\partial \lambda^{\mathcal{D}}}{\partial l_a}|_{\{d=0^+, l_\phi = l^*_\phi\}_{\phi=a,b}} \ = \ E_V[\frac{u_a - 1}{1 + l^*_a(u_a - 1)}] \ . \tag{15}$$

This equation is identical to (8) and the condition for l^*_b is (by definition) the same of eq. (8): this allows to define the probability Q (10) with the same properties of the previous section.

The equation specifying d^* is

$$0 \ = \ \frac{\partial \lambda^{\mathcal{D}}}{\partial d}|_{\{d=0^+, l_\phi = l^*_\phi\}_{\phi=a,b}} \ = \ E_V[\frac{f(u_a) - 1}{1 + l^*_a(u_a - 1)}] \ ,$$

which, using eq. (10) and the definition of f (see eq. (13), can be rewritten as

$$\mathcal{A}_0^{(1)} = E_Q[\mathcal{A}_T^{(1)}(u_a)] = E_Q[(S_0^a u_a - K)^+] \; . \tag{16}$$

Strictly speaking the NASA Principle gives the lower bound of $\mathcal{A}_0^{(1)}$ ask price since, for all option prices greater than this, no money will be invested in the *synthetic* option. However in a perfect competition market, the market maker is forced to price according to this lower bound.

Mutatis mutandis, a *synthetic option* $\mathcal{B}^{(1)}$ can be sold by the market maker when the speculator sells the underlying. In this case we obtain

$$\mathcal{B}_0^{(1)} = E_Q[\mathcal{B}_T^{(1)}(u_b)] = E_Q[(S_0^b u_b - K)^+] \; . \tag{17}$$

Defining

$$\begin{array}{l} \tilde{S}_T^a = S_0^a u_a \\ \tilde{S}_T^b = S_0^b u_b \end{array} \; , \tag{18}$$

which are martingale processes w.r.t. Q, we can rewrite eq. (16) and (17) as

$$\begin{array}{l} \mathcal{A}_0^{(1)} = E_Q[(\tilde{S}_T^a - K)^+] \\ \mathcal{B}_0^{(1)} = E_Q[(\tilde{S}_T^b - K)^+] \end{array} \; , \tag{19}$$

proving in the *synthetic* option case the claim of eq. (1): prices are the expected values of the final payoffs even when transaction costs are present.

To conclude this section, let us consider the price of a *synthetic option* with physical delivery $\mathcal{A}^{(2)}$ that expires after 2 trades of the portfolio with only the underlying and the bank account. This *synthetic option* can be bought at the m^{th} transaction and expires at the $(m+2)^{th}$.

In table **2** we report the payoffs.

$(m+1)^{th}trade$	$(m+2)^{th}trade$	*Payoff at the maturity*
ask	*ask*	$\mathcal{A}_T^{(2)} \equiv (S_T^a - K)^+ = (S_0^a u_{aa} u_{aa} - K)^+$
ask	*bid*	$\mathcal{A}_T^{(2)} \equiv (S_T^b - K)^+ = (S_0^a u_{aa} u_{ab} - K)^+$
bid	*ask*	$\mathcal{A}_T^{(2)} \equiv (S_T^a - K)^+ = (S_0^a u_{ab} u_{ba} - K)^+$
bid	*bid*	$\mathcal{A}_T^{(2)} \equiv (S_T^b - K)^+ = (S_0^a u_{ab} u_{bb} - K)^+$

Table **2**: Payoff of *synthetic options* $\mathcal{A}^{(2)}$. We report the $(m+1)^{th}$ and the$(m+2)^{th}$ *state* of the transaction on the underlying.

This *synthetic* option can be traded even at the $(m+1)^{th}$ transaction. It will be bought if the speculator acquires the underlying, sold otherwise; in fact, in the NASA Principle we compare speculator's optimal trading strategy with and without the derivative and, since a call option is an increasing function of the underlying, it is obviously not optimal for a speculator to buy the underlying and at the same time sell the call option (and vice versa).

Imposing the NASA Principle and following at each trade the same optimization procedure, we obtain

$$\mathcal{A}_0^{(2)} = Q_{aa}\mathcal{A}^{(1)} + Q_{ab}\mathcal{B}^{(1)} = E_{Q_a^{(2)}}[(\tilde{S}_T^a - K)^+] \ , \tag{20}$$

where we have defined the probability distribution $Q_a^{(2)}(\tilde{S}_T|\tilde{S}_0)$ as the convolution of 2 'one-time-step' probability Q (starting from an ask) and

$$\tilde{S}_T^a = S_0 u_a^{(1)} u^{(2)} \tag{21}$$

which is martingale processes w.r.t. $Q_a^{(2)}$.

In the same way, we obtain the price of a *synthetic* option that expires after N trades as

$$\begin{array}{rl} \mathcal{A}_0^{(N)} &= E_{Q_a^{(N)}}[(\tilde{S}_T^a - K)^+] \\ \mathcal{B}_0^{(N)} &= E_{Q_b^{(N)}}[(\tilde{S}_T^b - K)^+] \end{array} \ , \tag{22}$$

where $Q^{(N)}$ is the N^{th} convolution of Q and

$$\begin{array}{rl} \tilde{S}_T^a &= S_0^a U_{aa}^{(N)} \\ \tilde{S}_T^b &= S_0^a U_{ab}^{(N)} \end{array} \ , \tag{23}$$

with $U_{\phi\psi}^{(N)}$ the product of all 'one-time-step' asset returns u with a starting *state* ϕ at time 0 and a *state* ψ at the maturity T. In appendix **B** it is deduced the value of $Q^{(N)}$ in the different cases.

In the next section we show how eq. (22) is the fundamental ingredient of option pricing in presence of transaction costs.

4 Call option prices

In this section we show how prices of call options with physical delivery can be obtained via a proper combination of *synthetic* options.

As already underlined in the introduction, we first consider the more intuitive case when trading can be realized only after finite variations of the underlying. In figure **1**, we show a possible evolution of underlying price during the life of an option. Let us analyze in detail this figure.

The speculator can buy the option at τ_0, the same time he trades on the underlying. Since in an optimal strategy, a speculator trades only simultaneously on the underlying and the derivative[1] three more transactions can take place before the maturity of the derivative contract. Thus, up to time τ_3, this option is equivalent to a *synthetic* option $\mathcal{A}^{(3)}$, where in the final payoff of this option we have to consider S_{τ_3}, the underlying *value* at time τ_3. However, in option pricing, the last lag between τ_3 and T is irrelevant. In fact, as we have stressed in section **2**, in the optimal trading strategy, the average time between two transactions is almost surely zero and then the underlying *value* is almost surely $S_{\tau_3} = S_T$. Finally, to apply the NASA Principle, we have to specify how the *roll over* is realized; similarly to the previous section, we assume for notational simplicity that at each time there is an option with the same time-to-maturity and moneyness. The speculator will trade again acquiring or selling a new call option with same time-to-maturity and moneyness at the first transaction after the expiration of the previous option (τ_4 in figure **1**).

The path described in figure **1** is one of the possibilities. Call option price is then a combination of *synthetic options* N, each one weighted with the probability $\mathcal{P}(N)$ to have

[1] It is straightforward to prove that the optimal timing in the management of a a portfolio with (behind the risk-less asset) a risky asset with a return which is a strictly increasing function of (only) underlying return is exactly the same of a portfolio with the underlying. Call option is, of course, a strictly increasing function of underlying return but it also decreases its value with time; however, in the problem described in this paper the decrease in option-time-value plays no role since, in the optimal trading strategy, the average time between two trades is zero.

N trades in a lag T

$$C_0^a = \sum_{N=0}^{\infty} \mathcal{P}_a(N) Q_a^{(N)}[(\tilde{S}_T^a - K)^+] \ , \qquad (24)$$

where $\mathcal{P}_a$ is the probability $\mathcal{P}$ when beginning with a state a.

Plain vanilla call option prices are then obtained choosing the optimal values for the barriers. Since, the speculator has to include almost no options in his portfolio according to the NASA Principle, the optimal barriers will be the same of the portfolio without options (i.e. $\delta = 0$ and Δ as in eq. (12)). Taking the optimal values for the barriers we then obtain the ask call price

$$C_0^a = \int_0^\infty dy \mathcal{P}_a(y) \int_0^1 dx [\mathcal{Q}_{aa}(x|y)(S_0 \mathcal{U}_{aa} - K)^+ + \mathcal{Q}_{ab}(x|y)(S_0 \mathcal{U}_{ab} - K)^+] \ , \qquad (25)$$

and in the same way the bid price

$$C_0^b = \int_0^\infty dy \mathcal{P}_b(y) \int_0^1 dx [\mathcal{Q}_{ba}(x|y)(S_0 \mathcal{U}_{ba} - K)^+ + \mathcal{Q}_{bb}(x|y)(S_0 \mathcal{U}_{bb} - K)^+] \ , \qquad (26)$$

where for $\mathcal{P}_\phi$ we consider its cumulant expansion up to the forth cumulant

$$\mathcal{P}_\phi(y) = \mathcal{N}(\mathcal{M}_\phi, \mathcal{V}_\phi) \left\{ 1 + \frac{\mathcal{S}_\phi \mathcal{V}_\phi^3}{3!} H_3 \left(\frac{y - \mathcal{M}_\phi}{\mathcal{V}_\phi} \right) + \frac{\mathcal{K}_\phi \mathcal{V}_\phi^4}{4!} H_4 \left(\frac{y - \mathcal{M}_\phi}{\mathcal{V}_\phi} \right) \right\} \ ,$$

with $\mathcal{N}$ the normal distribution, H_m the m^{th} Hermite function; y's mean value $\mathcal{M}$, standard deviation $\mathcal{V}$, Skewness $\mathcal{S}$ and Kurtosis $\mathcal{K}$ can be found in appendix **C** (see eq. (33) for the symmetric case). The values of $\mathcal{Q}_{\phi\psi}(x|y)$ and $\mathcal{U}_{\phi\psi}$ are

ϕ	ψ	$\mathcal{Q}_{\phi\psi}(x\|y)$	$\mathcal{U}_{\phi\psi}$
a	a	$e^{-y\alpha/\Delta^2}\delta(x) + e^{-y[\alpha+\Delta x]/\Delta^2} \sum_{n=1}^{\infty} \frac{x^{n-1}(1-x)^n}{n!(n-1)!} \frac{y^{2n}\alpha^n\beta^n}{\Delta^{4n}}$	$e^{y(2x-1)/\Delta}$
a	b	$e^{-y[\alpha+\Delta x]/\Delta^2} \sum_{n=0}^{\infty} \frac{x^n(1-x)^n}{(n!)^2} \frac{y^{2n+1}\alpha^{n+1}\beta^n}{\Delta^{4n+2}}$	$e^{y(2x-1)/\Delta+\Delta}$
b	a	$e^{-y[\alpha+\Delta x]/\Delta^2} \sum_{n=0}^{\infty} \frac{x^n(1-x)^n}{(n!)^2} \frac{y^{2n+1}\alpha^n\beta^{n+1}}{\Delta^{4n+2}}$	$e^{y(2x-1)/\Delta-\Delta}$
b	b	$e^{-y\beta/\Delta^2}\delta(1-x) + e^{-y[\alpha+\Delta x]/\Delta^2} \sum_{n=1}^{\infty} \frac{x^n(1-x)^{n-1}}{n!(n-1)!} \frac{y^{2n}\alpha^n\beta^n}{\Delta^{4n}}$	$e^{y(2x-1)/\Delta}$

(27)

with $\delta(\cdot)$ Dirac's delta and

$$\alpha = \frac{\Delta}{e^\Delta - 1} \qquad \beta = \frac{\Delta}{1 - e^{-\Delta}} \ .$$

and Δ given by eq. (12).

It is straightforward to show that eq. (25) and (26) reduces to the Black-Scholes formula in the limit for $\gamma = 0$ (and then $\Delta = 0$).

In figure **2**, we plot call bid price and call bid-ask spread. In particularly we observe that bid-ask spread for options tends to zero for options deeply out-of-the-money and to $exp(\gamma) - 1$ for options deeply in-the-money, similar to what has already been observed by Davis et Al. [9].

5 Conclusions

We have shown how to price call options with physical delivery in presence of transaction costs. Ask and bid prices are the expected values of the final payoff.

Eq. (26) and (25) are closed formulas for call option prices. Black-Scholes results are obtained in the limit of absence of transaction costs and bid-ask spread is the same of Davis et Al. [9] for options deeply in- and out-of-the-money.

We have shown how these prices are obtained in a game between a market maker in a perfect competition market and a speculator who optimizes the growth rate of his capital. A question that naturally arises is whether one obtains different prices depending on different risk-adverse behavioral assumption on investors.

It can be shown [3] that the same results hold for investors maximizing an utility in the HARA class

$$\mathcal{U}(W) = \frac{W^\eta}{\eta} \quad , \quad \eta < 0 \quad ,$$

since the timing of an optimal trading rule of a portfolio without derivative is the same (and then the same probabilities $\mathcal{P}$ and $\mathcal{Q}$) of the "game" described in this paper.

Appendix A

In this appendix we summarize the results of the optimal portfolio selection in absence of derivative [2]. In table **A.1** we have recall the results in the case of generic barriers δ

and Δ. In particular the probability to reach a barrier and the average exit times can be obtained using elementary probability theory [6].

ask	*bid*
$\pi_a = \dfrac{1-e^{-\frac{2\mu}{\sigma^2}\delta}}{1-e^{-\frac{2\mu}{\sigma^2}(\delta+\Delta+\gamma)}}$	$\pi_b = \dfrac{e^{\frac{2\mu}{\sigma^2}\delta}-1}{e^{\frac{2\mu}{\sigma^2}(\delta+\Delta+\gamma)}-1}$
$\mathcal{T}_a = \dfrac{1}{\mu}[\delta-(\delta+\Delta+\gamma)\pi_a]$	$\mathcal{T}_b = \dfrac{1}{\mu}[-\delta+(\delta+\Delta+\gamma)\pi_b]$
$l_a^* = \dfrac{\pi_a}{1-\exp(-\delta)} - \dfrac{1-\pi_a}{\exp(\Delta)-1}$	$l_b^* = \dfrac{1-\pi_b}{1-\exp(-\Delta)} - \dfrac{\pi_b}{\exp(\delta)-1}$

Table A.1: Generic barriers (Δ and δ): probability of to reach the upper barrier after an ask (π_a) and the lower barrier after a bid (π_b); average time between two trades ($\mathcal{T}$) and optimal capital fraction to be invested in the risky asset (l^*) in the ask and bid case. In $\mu = 0$ all the quantities are defined by continuity.

Let us notice that the average exit time goes to zero linearly with δ for small δ, or, said differently, in the optimal solution the average time between two transactions is almost surely zero. After having chosen the optimal values for the barriers, the optimal capital fraction to be invested in the risky asset is

ask	*bid*
$l_a^{**} = \dfrac{2\mu/\sigma^2}{1-\exp(-\frac{2\mu}{\sigma^2}(\Delta+\gamma))} - \dfrac{1}{\exp(\Delta)-1}$	$l_b^{**} = \dfrac{1}{1-\exp(-\Delta)} - \dfrac{2\mu/\sigma^2}{\exp(\frac{2\mu}{\sigma^2}(\Delta+\gamma))-1}$

where Δ is given by eq. (12). In $\mu = 0$ all the quantities are defined by continuity.

The optimal growth rate of a portfolio with a bank account and a risky asset in presence of transaction costs is

$$\lambda^{**} = \frac{\sigma^2}{2} l_a^{**} l_b^{**} \ . \tag{28}$$

Appendix B

In this appendix we derive the probabilities $Q^{(N)}$ as the convolutions of the 'one-time-step' probabilities Q. This distribution depends on the initial *state* ϕ and on the final one (at the maturity) ψ

$$Q_{\phi\psi}^{(N)} = \mathcal{N}_{\phi\psi}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb}) Q_{aa}^{n_{aa}} Q_{ab}^{n_{ab}} Q_{ba}^{n_{ba}} Q_{bb}^{n_{bb}}$$

where $\mathcal{N}_{\phi\psi}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb})$ is the number of paths that starting from ϕ lead to ψ in N steps:

$$\begin{aligned}
\mathcal{N}_{aa}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb}) &= \frac{n_b!(n_a-1)!n_{ba}}{n_{aa}!n_{ab}!n_{ba}!n_{bb}!} \\
\mathcal{N}_{ab}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb}) &= \frac{n_b!(n_a-1)!n_{ab}}{n_{aa}!n_{ab}!n_{ba}!n_{bb}!} \\
\mathcal{N}_{ba}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb}) &= \frac{(n_b-1)!n_a!n_{ba}}{n_{aa}!n_{ab}!n_{ba}!n_{bb}!} \\
\mathcal{N}_{bb}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb}) &= \frac{(n_b-1)!n_a!n_{ab}}{n_{aa}!n_{ab}!n_{ba}!n_{bb}!}
\end{aligned} \quad . \tag{29}$$

where $n_a = n_{aa} + n_{ab}$ and $n_b = n_{ba} + n_{bb}$.

The number of paths can be obtained via standard combinatorial analysis. Let us derive for example $\mathcal{N}_{aa}(N; n_{aa}, n_{ab}, n_{ba}, n_{bb})$. In this case the path starts with $\phi = a$ and ends with $\psi = a$ and for each couple of characters there are n_{aa}, n_{ab}, n_{ba} and n_{bb} couples. We divide these couples in three sets: in the first we put a couple (ba), in the second one all the couples starting with b except the one in the first set (number of couples in this set: $n_b - 1 = n_{ba} + n_{bb} - 1$) and in the third one all the couples starting with a (number of couples in this set: $n_a = n_{aa} + n_{ab}$). A generic chain starting with a and ending with a can be obtained as follows: select (without replacement) a couple from the second set; if the last character of the couple is an a continue to extract in the same set and in the third set otherwise... and so on. When there no more couples in the second set take the couple in the first set: since $n_{ab} = n_{ba} \equiv n$ we are sure to end with an a. We have n_{ba} ways to select the couple of the first set, $(n_b - 1)!$ ways to select (without replacement) couples from the second set and $n_a!$ from the third set. The result is obtained after observing that we are interested in the paths that are not equal after permutations. The other cases can be derived following a similar procedure.

In the limit δ to zero we obtain the $\mathcal{Q}$s of eq. (27) defining α and β s.t.

$$\begin{aligned}
Q_{ab} &= \tfrac{\alpha}{\Delta}\delta + O(\delta^2) \\
Q_{ba} &= \tfrac{\beta}{\Delta}\delta + O(\delta^2)
\end{aligned}$$

and

$$x \equiv n_b/N \quad y \equiv N\delta\Delta \quad . \tag{30}$$

It is straightforward to prove that for every y $\mathcal{Q}$ is normalized and

$$\int dx \mathcal{Q}_{\phi a}(x|y)U_{\phi a} + \mathcal{Q}_{\phi b}(x|y)U_{\phi b} = 1 \quad , \quad \phi = a, b \quad ,$$

with $U_{\phi\psi}$ defined in eq. (27).

Appendix C

In this section we derive the probability $\mathcal{P}_\phi(N;T)$ to exit N times from the barriers in a lag T starting from a state ϕ. As described in the paper, the barriers for log-prices are $\Delta + \gamma$ and $-\delta$ if the previous transaction is an ask, δ and $-\Delta - \gamma$ after a bid. We first consider the symmetric case ($\mu = 0$), showing then how to generalize the results to the case with drift.

Symmetric case

In the case with no drift, due to the symmetry of the problem $\mathcal{P}_\phi(N;T)$ does not depend on the initial state ϕ. Let us define $p(t)$ the probability distribution to reach one of the two barriers at time t (exit time). The exit time distribution $p(t)$ satisfies a Fokker-Planck equation, which can be solved via the Fourier transform [6]

$$\tilde{p}(\nu) = \frac{\sin \frac{\delta}{\sigma^2}\sqrt{-2i\nu\sigma^2}}{\sin \frac{\Delta+\gamma+\delta}{\sigma^2}\sqrt{-2i\nu\sigma^2}}$$

where the Fourier transform $\Re^+ \to \Re$ is defined as

$$\tilde{p}(\nu) = \int_0^\infty dT \ \mathrm{e}^{-i\nu t} p(t)$$
$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\nu \ \mathrm{e}^{i\nu t} \ \tilde{p}(\nu)$$

The distribution $\mathcal{P}(N;T)$ can be derived as in a standard renewal problem [6]. By definition

$$\mathcal{P}(N;T) \equiv \int_{\sum_{i=1}^N t_i \leq T} dt_1 \ldots dt_N \left(\prod_{i=1}^N p(t_i) \right) \left(1 - \int_0^{T-\sum_{i=1}^N t_i} dt_{N+1} p(t_{N+1}) \right)$$

and then

$$\mathcal{P}(N;T) = \mathcal{R}(N;T) - \mathcal{R}(N+1;T)$$

where

$$\mathcal{R}(N;T) \equiv \int_{\sum_{i=1}^{N} t_i \leq T} dt_1 \dots dt_N \left(\prod_{i=1}^{N} p(t_i) \right)$$

is the probability distribution to exit *at least* N times in the lag T.

$\mathcal{R}(N;T)$ can be rewritten as:

$$\mathcal{R}(N;T) = \int_0^T dt_1 \; p(t_1) \int_{\sum_{i=2}^{N} t_i \leq T - t_1} dt_2 \dots dt_N \left(\prod_{i=2}^{N} p(t_i) \right) = \int_0^T dt_1 \; p(t_1) \; \mathcal{R}_{N-1}(T - t_1) \tag{31}$$

and $\mathcal{R}(0;T) = 1$.

Using the Fourier transform we obtain

$$\tilde{\mathcal{R}}(N;\nu) = \tilde{p}(\nu) \; \tilde{\mathcal{R}}(N-1;\nu)$$

and then

$$\tilde{\mathcal{R}}(N;\nu) = \tilde{p}(\nu)^N \; \tilde{\mathcal{R}}(0;\nu) \quad ,$$

By definition the initial value is

$$\tilde{\mathcal{R}}(0;\nu) = \int_0^\infty dT \; \mathrm{e}^{-i\nu t} \quad .$$

Thus

$$\tilde{\mathcal{P}}(N;\nu) = \tilde{p}(\nu)^N \; (1 - \tilde{p}(\nu)) \tilde{\mathcal{R}}(0;\nu)$$

or equivalently

$$\mathcal{P}(N;T) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\nu \; \mathrm{e}^{i\nu T} \; \tilde{p}(\nu)^N \; \frac{1 - \tilde{p}(\nu)}{\nu - i\epsilon} \tag{32}$$

which can be integrated for every N using Jordan's Lemma [14].

We are interested in the limit for δ which goes to zero. Even if we are not able to solve eq. (32) in this case, we can derive the leading term (considering the residual in $\nu = 0$) of $\mathcal{P}(y;T)$ cumulants and then consider their limit for δ equal to zero, where we have defined $y \equiv N\Delta\delta$ as in eq. (30).

The first four cumulants of $\mathcal{P}(y;T)$ are

$$\begin{aligned}
\mathcal{M} &= \frac{(\Delta+\gamma)^2}{6} + \sigma^2 T \\
\mathcal{V} &= \frac{(\Delta+\gamma)^4}{180} + \frac{(\Delta+\gamma)^2\sigma^2 T}{3} \\
\mathcal{S} &= \frac{2\sqrt{5}(\Delta+\gamma)(11(\Delta+\gamma)^2+72\sigma^2 T)}{((\Delta+\gamma)^2+60\sigma^2 T)^{3/2}} \\
\mathcal{K} &= \frac{6(\Delta+\gamma)^2(3839(\Delta+\gamma)^2+3360\sigma^2 T)}{7((\Delta+\gamma)^2+60\sigma^2 T)^2}
\end{aligned} \tag{33}$$

A cumulant expansion of $\mathcal{P}(y;T)$ is well justified by the fact that Δ goes to zero in the Black-Scholes limit and the distribution of y reduces to a Dirac's delta around $\sigma^2 T$ in this limit.

General case

In the general case we can obtain the exit time probability $p_{\phi\psi}(t)$ starting from a *state* ϕ and ending with a *state* ψ, solving the Fokker-Planck equation with drift. Their Fourier transforms are

$$\tilde{p}_{aa}(\nu) = e^{-\mu\delta/\sigma^2} g(\nu), \qquad \tilde{p}_{ab}(\nu) = e^{\mu(\Delta+\gamma)/\sigma^2} f(\nu)$$
$$\tilde{p}_{ba}(\nu) = e^{-\mu(\Delta+\gamma)/\sigma^2} f(\nu), \qquad \tilde{p}_{bb}(\nu) = e^{\mu\delta/\sigma^2} g(\nu)$$

with

$$g(\nu) \equiv \frac{\sin\frac{\Delta+\gamma}{\sigma^2}\sqrt{-\mu^2-2i\nu\sigma^2}}{\sin\frac{\Delta+\gamma+\delta}{\sigma^2}\sqrt{-\mu^2-2i\nu\sigma^2}} \quad , \quad f(\nu) \equiv \frac{\sin\frac{\delta}{\sigma^2}\sqrt{-\mu^2-2i\nu\sigma^2}}{\sin\frac{\Delta+\gamma+\delta}{\sigma^2}\sqrt{-\mu^2-2i\nu\sigma^2}}$$

In the case with drift the relation equivalent to eq. (31) is

$$\mathcal{R}_a(N,T) = \int_0^T dt\ p_{aa}(t)\ \mathcal{R}_a(N-1,T-t) + \int_0^T dt\ p_{ba}(t)\ \mathcal{R}_b(N-1,T-t)$$
$$\mathcal{R}_b(N,T) = \int_0^T dt\ p_{ab}(t)\ \mathcal{R}_a(N-1,T-t) + \int_0^T dt\ p_{bb}(t)\ \mathcal{R}_b(N-1,T-t)$$

or equivalently in Fourier space

$$\begin{pmatrix} \tilde{\mathcal{R}}_a(N;\nu) \\ \tilde{\mathcal{R}}_b(N;\nu) \end{pmatrix} = \begin{pmatrix} \tilde{p}_{aa}(\nu) & \tilde{p}_{ba}(\nu) \\ \tilde{p}_{ab}(\nu) & \tilde{p}_{bb}(\nu) \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{R}}_a(N-1;\nu) \\ \tilde{\mathcal{R}}_b(N-1;\nu) \end{pmatrix}$$

with $\tilde{\mathcal{R}}_{a,0}(\nu) = \tilde{\mathcal{R}}_{b,0}(\nu) = \tilde{\mathcal{R}}_0(\nu)$.

We then obtain

$$\tilde{\mathcal{P}}_a(N;\nu) = \left[\xi_1(\nu)\lambda_1^N(\nu)(1-\lambda_1(\nu)) + \xi_2(\nu)\lambda_2^N(\nu)(1-\lambda_2(\nu))\right]\frac{\tilde{\mathcal{R}}(0;\nu)}{2\rho(\nu)}$$

and

$$\tilde{\mathcal{P}}_b(N;\nu) = \left[\eta_1(\nu)\lambda_1^N(\nu)(1-\lambda_1(\nu)) + \eta_2(\nu)\lambda_2^N(\nu)(1-\lambda_2(\nu))\right] \frac{\tilde{\mathcal{R}}(0;\nu)}{2\rho(\nu)}$$

where

$$\rho(\nu) = \sqrt{\sinh^2\left(\frac{\mu\delta}{\sigma^2}\right) g^2(\nu) + f^2(\nu)}$$

$$\eta_1(\nu) = \sinh\left(\frac{\mu\delta}{\sigma^2}\right) g(\nu) + e^{-\mu(\Delta+\gamma)/\sigma^2} f(\nu) + \rho(\nu)$$

$$\eta_2(\nu) = -\sinh\left(\frac{\mu\delta}{\sigma^2}\right) g(\nu) - e^{-\mu(\Delta+\gamma)/\sigma^2} f(\nu) + \rho(\nu)$$

$$\xi_1(\nu) = -\sinh\left(\frac{\mu\delta}{\sigma^2}\right) g(\nu) + e^{\mu(\Delta+\gamma)/\sigma^2} f(\nu) + \rho(\nu)$$

$$\xi_2(\nu) = \sinh\left(\frac{\mu\delta}{\sigma^2}\right) g(\nu) - e^{\mu(\Delta+\gamma)/\sigma^2} f(\nu) + \rho(\nu)$$

$$\lambda_1(\nu) = \cosh\left(\frac{\mu\delta}{\sigma^2}\right) g(\nu) + \rho(\nu)$$

$$\lambda_2(\nu) = \cosh\left(\frac{\mu\delta}{\sigma^2}\right) g(\nu) - \rho(\nu)$$

Finally $\mathcal{P}_b(y;T)$ and $\mathcal{P}_b(y;T)$ can be obtained as in the symmetric case.

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Figure legends

Figure 1 A possible dynamics of underlying prices during the life of a derivative contract. We plot log-prices *vs.* time. Asset *value* (continuous line) is equal to ask price after an ask and to bid price after a bid; the dashed line represents the other price.

Figure 2 Call bid price (continuous line and left axis) and bid-ask spread (dashed line and right axis) for an option of maturity 3 years, strike price $K = 1$, volatility 70 percent, $\mu = 0$ and $\gamma = 1.126 \cdot 10^{-3}$.

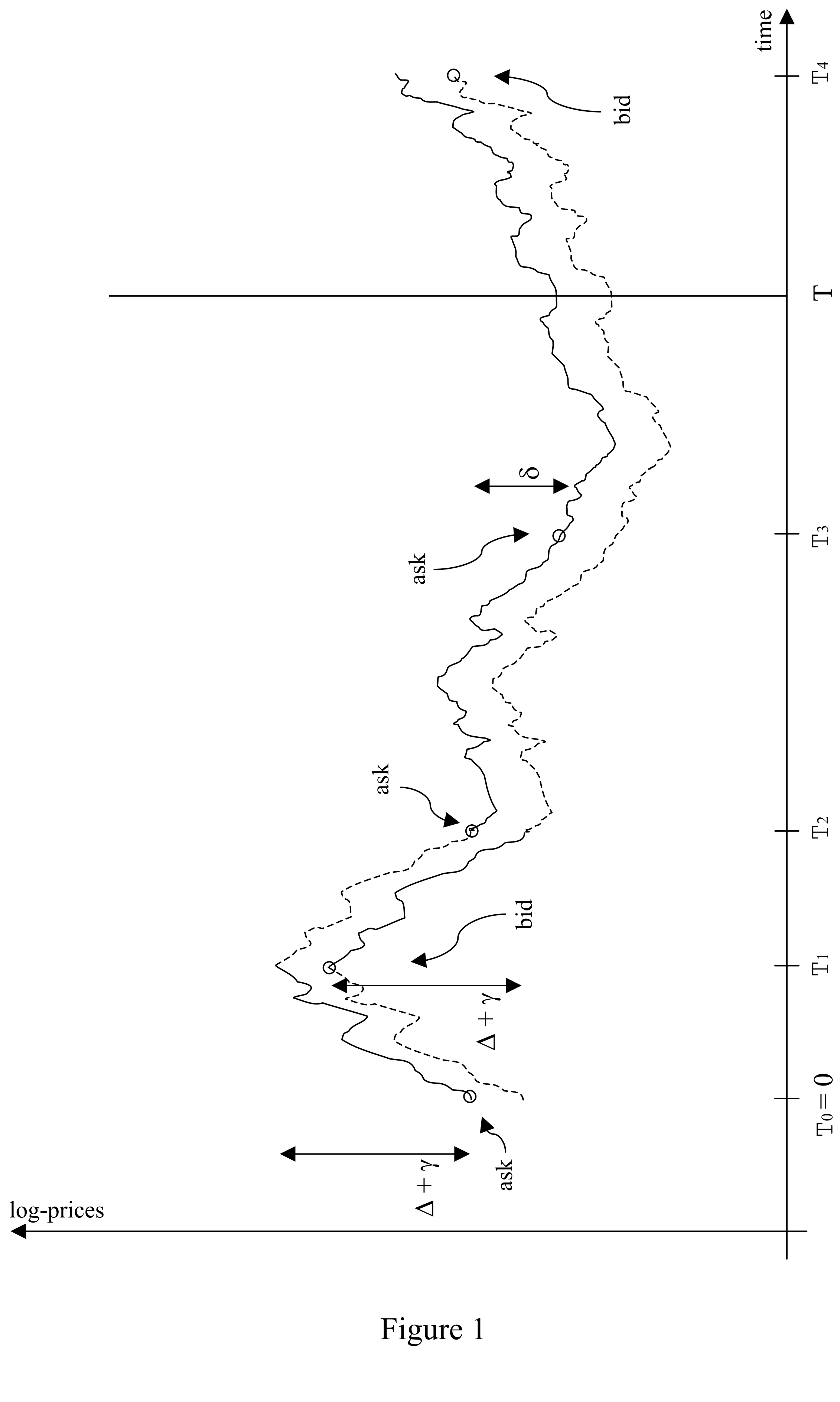

Figure 1

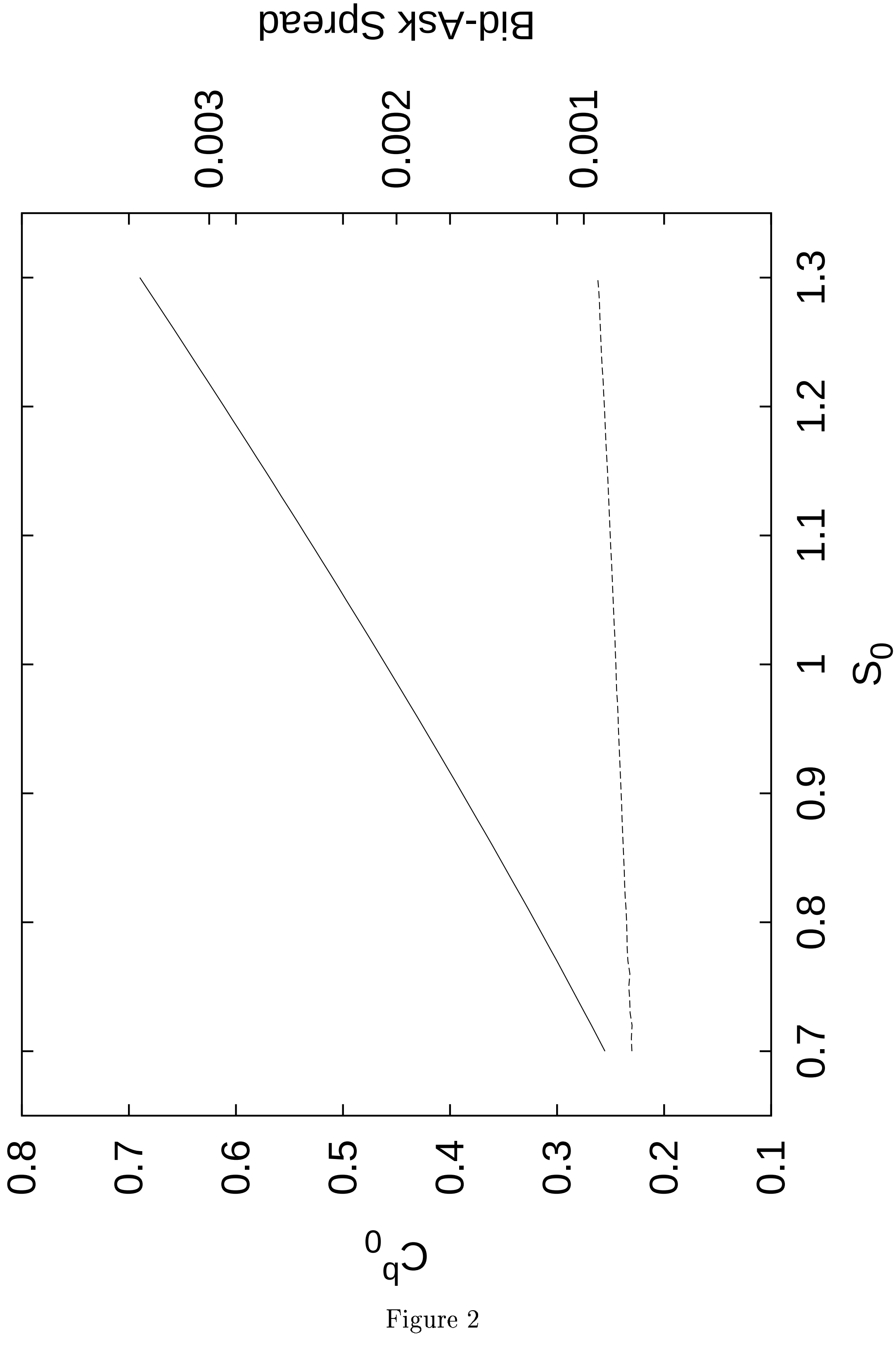
Bid-Ask Spread
0.003
0.002
0.001
0.7
0.8
0.9
1
1.1
1.2
1.3
S_0
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
C^b_0

Figure 2