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Physica A 280 (2000) 566–581

PHYSICA A

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Markovian approximation in foreign exchange markets

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Received 25 January 2000

Abstract

In this paper, using the exit-time statistic, we study the structure of the price variations for the high-frequency data set of the bid–ask Deutschemark/US dollar exchange rate quotes registered by the inter-bank Reuters network over the period October 1, 1992 to September 30, 1993. Having rejected random-walk models for the returns, we propose a Markovian model which reproduce the available information of the financial series. Besides the usual correlation analysis we have verified the validity of this model by means of other tools all inspired by information theory. These techniques are not only severe tests of the approximation but also evidence of some aspects of the data series which have a clear financial relevance.

PACS: F31

Keywords: Exit times; Foreign exchange markets; Markov process

1. Introduction

Any financial theory begins introducing a reasonable model for price variation in terms of a suitable stochastic process. In this paper we want to select an asset-pricing model able to describe some features interesting from a financial point of view at *weak* level, i.e., including only information arising from analysis of historical prices and not including other public [1–4] or private [5] information.

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0378-4371/00/$ - see front matter
PII: S0378-4371(00)00094-7

To attain such an aim we analyze the foreign exchange market because it presents several advantages compared with other financial markets. First of all it is a very liquid market. This feature is important because every financial market provides a single sample path, the set of the registered asset quotes. To be sure that a statistical description makes sense, a reasonable requirement is that this (unique) sequence of quotes is not dominated by single events or single trader's operations. A natural candidate for such (if possible) description is then a liquid market involving several billions of dollars, daily traded by thousand of actors. In addition, the foreign exchange market has no business time limitations. Many market makers have branches worldwide so trading can occur *almost* continuously. So in the analysis one can avoid to consider problems involved in the opening and closure of a particular market, at least as a first approximation. Finally, if one considers the currency exchange, the returns

$$r_t = \ln \frac{S_{t+1}}{S_t} \tag{1}$$

are almost symmetrically distributed, where S_t is the price at the time t defined as the average between bid and ask prices.

In this paper we investigate the possibility to describe the Deutschemark/US dollar exchange (the most liquid market) in term of a Markov process. We consider a high-frequency data set to have statistical relevance of the results. Our data, made available by Olsen and Associated, contains all worldwide 1,472,241 bid–ask Deutschemark/US dollar exchange rate quotes registered by the inter-bank Reuters network over the period October 1, 1992 to September 30, 1993.

One of the main problems when analyzing financial series is that the quotes are irregularly spaced. In Section 2, we briefly describe some different ways to introduce time in finance, and we discuss why we choose the *business* time, i.e., the time of a transaction is its position in the sequence of the registered quotes.

The history of the efforts in the proposal of proper stochastic processes for price variations is very long. An efficient foreign currency market, i.e., where prices reflect the whole information, suggests that returns are independently distributed. Following Fama [6] we shall call hereafter "random walk" a financial model where returns are independent variables. Without entering into a detailed review we recall the seminal work of Bachelier [7] who assumed (and tested) that price variations follow an independent Gaussian process. Now it is commonly believed that returns do not behave according to a Gaussian. Mandelbrot [8] has proposed that returns are Levy-stable distributed, still remaining independent random variables. A recent proposal is the truncated Levy distribution model introduced by Mantegna and Stanley [9] which well fits financial data, even considering them at different time lags.

Since the financial importance of correlations in order to detect arbitrage opportunities (see Ref. [10] for a review and Ref. [11] for more recent results) it is essential to introduce a non-questionable technique able to answer to the question: can "random walk" models correctly describe return variations?

In Section 2, we show that "random walk" is inadequate to describe even qualitatively some important features of price behavior. We measure the probability distribution of exit times, i.e., the lags to reach a given return amplitude Δ. This analysis not only shows the presence of strong correlations but also leads to a natural measure of time which is intrinsic of the market evolution, i.e., the time in which the market has such a fluctuation. Following Baviera et al. [11] we call this time Δ-trading time.

In Section 3, measuring the time in Δ-trading time, we discuss the validity of a Markovian approximation for the market evolution. First, using the quotes, we build a Markovian model of order m, then, starting from the usual correlation function approach, we consider several techniques to verify the quality of this description.

We also perform an entropic analysis, inspired by the Kolmogorov ε-entropy [12]. This kind of analysis is equivalent to consider a speculator who cares only of market fluctuation of a given size Δ [11].

Finally, we use other statistical tools to test the validity of such an approximation, namely the mutual information introduced by Shannon [13] and the Kullback entropy [14] which measures the "discrepancy" between the Markovian approximation and the "true" return process.

The validity of the Markovian approximation not only implies the failure of all that models with independent increments ("random walk"), but give quantitative indications about the market inefficiency (neglecting transaction costs).

In Section 4, we summarize and discuss the results attained.

2. Exit times

One of the main (and unsolved) problems in tick data analysis concerns the irregular spacing of quotes. There are several candidates to measure the time of each transaction.

The first one is obviously the *calendar* time, i.e., the Greenwich Meridian Time at which the transaction occurs. The trouble with such a choice is rather evident: there are periods of no transaction. The simplest way (see, e.g. Ref. [9]) to overcome this difficulty is to cut "nights" and "weekends" from the signal, i.e., assuming a zero time lag between the closure and reopening of the market. Of course in a worldwide series is less evident what does it mean "night" or "weekend", but it is easy to observe that during a day or a week there are lags when no transaction is present. An improvement of the above procedure is to rescale the *calendar* time with a measure of market activity, i.e., to create a new time scale under which in all lags the same market activity occurs. We do not enter here in the literature on this subject, we briefly mention the procedure in which one measures the market activity with the average absolute number of quotes per lag (e.g. of 15 min) or with the average absolute price change over each lag [15,16]. Let us note that the weight does not change too much the time (roughly it is between 0.5 and 2.5), therefore there is not a big difference with the naive procedure. A similar approach has been introduced by Zhang [17] where the time lags are rescaled with a properly defined instantaneous volatility.

A slightly different approach is the *business* time. One considers all transactions equivalent and the time of the transaction is simply given by its position in the sequence of quotes. In this section, we shall adopt the *business* time; it looks a reasonable choice when facing a worldwide sequence where lags of no transaction are often a consequence of the geographical position on the earth surface of the most important markets. However, we have to stress that, at least for some statistical properties, there are not qualitatively differences using the *business* or the *calendar* time.

In this section, we study the distribution of the exit times at a given resolution Δ. This analysis will allow us to show that the "random walk" models cannot be a reasonable description. We stress that for "random walk" we mean all that models with independent increments.

This technique will help us to understand how to analyze financial time series. Introducing

$$r_{t,t_0} \equiv \ln \frac{S_t}{S_{t_0}} , \tag{2}$$

where t_0 is the initial *business* time and $t > t_0$, we wait until t_1 such as

$$|r_{t_1,t_0}| \geqslant \Delta . \tag{3}$$

Now, starting from t_1 with the above procedure we obtain S_{t_2}, and so on. In this way we construct the successions of returns and exit times at given Δ:

$$\{\rho_1, \rho_2, \ldots, \rho_k, \ldots\}, \quad \text{where } \rho_k \equiv \ln \frac{S_{t_k}}{S_{t_{k-1}}} \tag{4}$$

and

$$\{\tau_1, \tau_2, \ldots, \tau_k, \ldots\}, \quad \text{where } \tau_k \equiv t_k - t_{k-1} , \tag{5}$$

where $|\rho_k| \geqslant \Delta$ by definition (see Eq. (3)), and τ_k is the time after which we have the kth fluctuation of order Δ in the price.

In the following we call k the Δ-trading time, i.e., we enumerate only the transactions at which a fluctuation Δ is reached. Since the distribution of the returns is *almost* symmetric, the threshold Δ has been chosen equal for both positive and negative values.

In Fig. 1 we show the evolution of the returns r_{t,t_k} till they reach a fluctuation Δ, where k is the Δ-*trading* time.

Let us now study $P_\Delta(\tau)$, the probability distribution function (PDF) of exit times (5) for a given size Δ. From the shape of this PDF one can have indication if a stochastic process can be considered a good candidate to model price variation.

In Fig. 2 we show that $P_\Delta(\tau)$ has a different shape for Δ smaller or larger than the typical transaction cost γ^{Typ}, where we define

$$\gamma_t \equiv \frac{1}{2} \ln \frac{S_t^{(\mathrm{ask})}}{S_t^{(\mathrm{bid})}} \simeq \frac{S_t^{(\mathrm{ask})} - S_t^{(\mathrm{bid})}}{2 S_t^{(\mathrm{bid})}}$$

the transaction cost at time t, whose distribution has a narrow peak around its typical value $\gamma^{\mathrm{Typ}} = 2.4 \times 10^{-4}$.

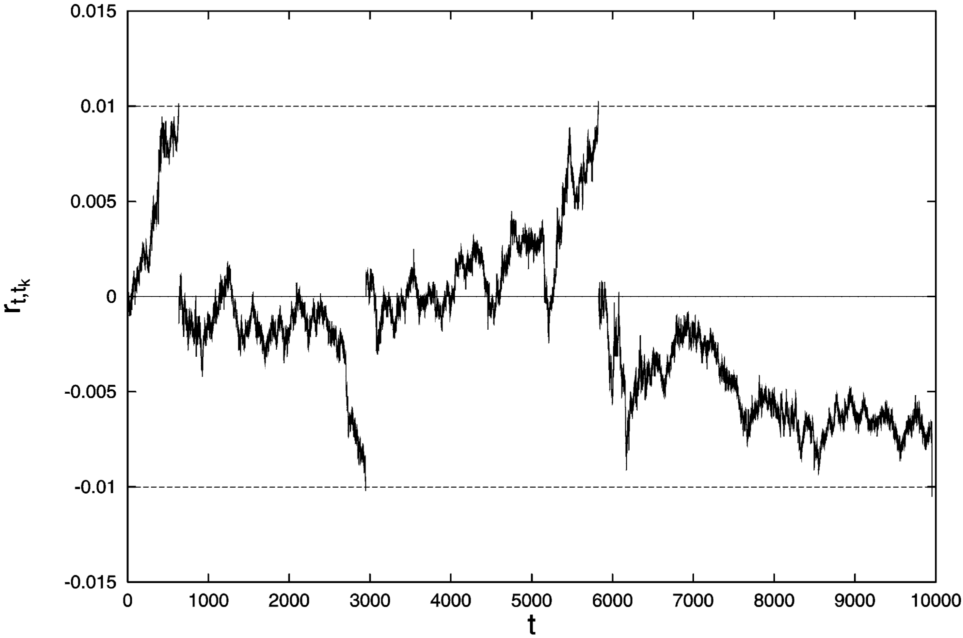

Fig. 1. Evolution of r_{t,t_k} with Δ=0.01. The Δ-*trading* time is zero (k=0) at the first transaction corresponding to 00 : 00 : 14 of October 1, 1992 in calendar time, and Δ-*trading* time is 4 ($k = 4$) at 11 : 59 : 28 of October 2, 1992 (9939 *business* time).

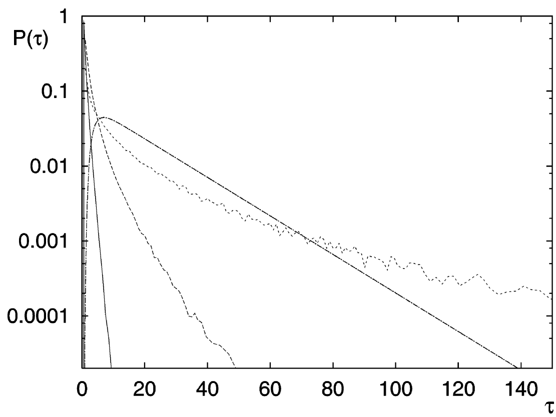

Fig. 2. Probability distribution function of the exit times τ_k for different Δ: $\Delta = 5 \times 10^{-5}$ (full line), 2×10^{-4} (dashed line) and 6×10^{-4} (dotted line). Those values are, respectively, smaller, similar to and larger the typical transaction costs. With the dashed–dotted line, we also show the PDF of the exit times for a Wiener process (see Eq. (6)) with the $\langle \tau \rangle = 16.85$ equal to that one obtained with $\Delta = 6 \times 10^{-4}$.

Note that $P_\Delta(\tau)$ is roughly exponential at small Δ, while for Δ larger than γ^{Typ}, $P_\Delta(\tau)$ clearly shows a non-exponential shape.

Let us now compare the PDF from the data analysis with the one obtained in "random walk" models. In the case of a Wiener process in absence of drift, following

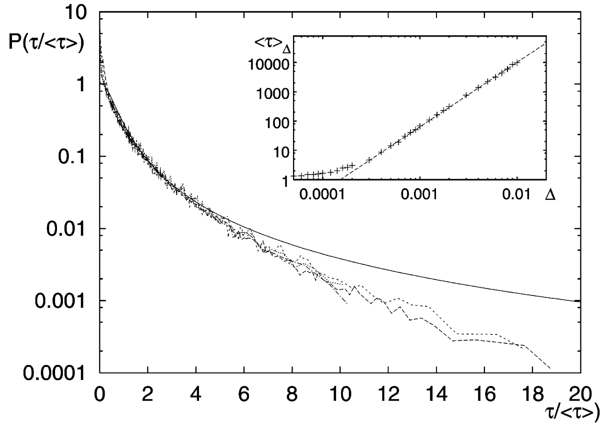

Fig. 3. Data collapse of $P_\Delta(\tau/\langle\tau\rangle)$ vs. $\tau/\langle\tau\rangle$. We have plotted four rescaled PDF: $\Delta = 4\times10^{-4}$, 6×10^{-4}, 8×10^{-4} and 1×10^{-3}. The full curve indicates the Cauchy distribution. In the inset we show $\langle\tau\rangle$ vs. Δ. The line shows the asymptotic behavior (for $\Delta > \gamma^{\mathrm{Typ}}$) of $\langle\tau\rangle \propto \Delta^\alpha$ with $\alpha = 2.2$.

Feller [18] one can find the exact solution for the PDF:

$$P_\Delta^{(W)}(\tau) = \frac{4}{\pi\hat{\tau}} \sum_{n=0}^{\infty} (-1)^n (2n+1) \mathrm{e}^{-(2m+1)^2\tau/\hat{\tau}}\,, \tag{6}$$

where $\hat{\tau} = (8\Delta^2)/(\pi^2\sigma^2)$ and σ^2 is the variance of the Wiener process. The main characteristics of this PDF are:

- It is peaked around $\langle\tau\rangle$. The similarity between average and typical value of this distribution allows us to interpret the average exit time as the one we expect to observe in the future with higher probability. This fact is not any more true in the case of a PDF with a power-law tail, and so it is more difficult to give a direct interpretation of the $\langle\tau\rangle$ shown in the inset in Fig. 3.
- It decreases exponentially at large τ. It is simple to understand that the exponential decay is true for any "random walk" model. In fact, in absence of correlations one has a finite probability, let us say less than q, to exit in a finite time $\hat{\tau}$. Therefore, the probability to exit after a time τ is less than $q^{\tau/\hat{\tau}}$, i.e., it is exponential. In Fig. 2 we also compare the normalized PDF computed from financial data with the $P_\Delta(\tau)$ of a "random walk" with the $\langle\tau\rangle$ equal to the one obtained from the signal with $\Delta = 6\times10^{-4}$. We recall that $\gamma^{\mathrm{Typ}} \simeq 2.4\times10^{-4}$.

A model built with independent variables is surely unable to reproduce $P_\Delta(\tau)$ and it is simple to understand that also processes with short memory, e.g. Markov process, give for the exit times a PDF qualitatively similar (i.e., decaying exponentially at large τ) to that one of a Wiener process.

Fig. 3 shows the PDF data collapse for different Δ, all greater than γ^{Typ}. We calculate the PDF of $\{\tau\}$ at different values of Δ, rescaled with the mean value $\langle\tau\rangle$:

$$P_\Delta\left(\frac{\tau}{\langle\tau\rangle}\right) \text{ vs. } \frac{\tau}{\langle\tau\rangle}\,.$$

We denote by $\langle\cdot\rangle$ the average of a sequence

$$\langle A\rangle \equiv \frac{1}{L}\sum_{l=1}^{L} A_l\,, \tag{7}$$

where L is the size of the sequence.

One observes that the rescaled PDF behave according to a single density function, which is well approximated, a part for large $\tau/\langle\tau\rangle$, by the Cauchy distribution

$$P_\Delta\left(\frac{\tau}{\langle\tau\rangle}\right) \approx \frac{2a}{\pi(a^2+(\tau/\langle\tau\rangle)^2)}\,. \tag{8}$$

The distribution $P_\Delta(\tau)$ shows a power-law tail up to an exponential cut (perhaps due to the finite size of the sample analyzed). We note that if Δ is larger than the typical transaction cost the probability distribution of τ has a very long tail and so the $\langle\tau\rangle_\Delta$ is not the typical value of the sequence (5).

In the inset we present $\langle\tau\rangle$ vs. Δ. One has a fairly clear scaling law of the average exit time as a function of Δ for more than three decades in $\langle\tau\rangle$:

$$\langle\tau\rangle_\Delta \sim \Delta^\alpha \quad \text{with} \quad \alpha \simeq 2.2 \tag{9}$$

for all Δ greater than the typical transaction costs γ^{Typ}. In the case of a Wiener process one has exactly $\langle\tau\rangle_\Delta \sim \Delta^2$.

The scaling behavior of $\langle\tau\rangle$ and the form of the distribution $P_\Delta(\tau)$ show the presence of strong correlations in the financial signal for $\Delta > \gamma^{\mathrm{Typ}}$, which force us to reject all the "random walk" models. Furthermore, the above analysis indicates the Δ-trading time as a natural candidate to measure time in financial context. In fact, this gives the intrinsic time for the market evolution to present a certain fluctuation, Δ (i.e., the Δ-trading time enumerates the times in which the market has such a fluctuation). In the next section we shall show that, measuring the time in such a way, a simple Markov process is a valid description of the market fluctuations.

3. The Markovian approximation

With the exit-time statistic one has a natural decomposition of the return series: the amplitude of return (which in general is near Δ (see Eq. (3))) and the sign of return. In this section we construct a Markov process of order m using a symbolic sequence obtained from the returns $\{\rho_k\}$ at fixed Δ. We use this process as a model for the dynamics of the return sign $z_k = \rho_k/|\rho_k|$. We stress that with other time measure (calendar or business time) this decomposition is impossible.

Symbolic dynamics is a rather powerful approach to catch the main statistical feature of time series. In order to construct a symbolic sequence from the succession $\{\rho_k\}$ we need a *coarse graining* procedure to partition the range data and then we assign a conventional symbol to each element of the partition. For our aims we use the most simple partition which is the same as to take the sign of returns at Δ-trading time. We perform the following transformation:

$$z_k = \begin{cases} -1 & \text{if } \rho_k < 0\,, \\ +1 & \text{if } \rho_k > 0\,. \end{cases} \tag{10}$$

The financial meaning of this codification is rather evident: the symbol -1 occurs if the stock price decreases of percentage Δ, while if the stock price increases of the same percentage the symbol is 1. In the following we indicate with $z^{(i)}$ the two possible values of the symbolic sequence.

Let us also explain the financial meaning of the return sequence $\{\rho_k\}$ at fixed Δ. A speculator, who modifies his portfolio only when a fluctuation of size Δ appears in the price sequence, cares only of these returns. Following Baviera et al. [11] we call *patient* investor such a speculator. He performs automatically a filtering procedure: he rejects all the quotes which do not change at least of a percentage Δ of the price. Therefore, this investor is interested only in the forecasting the sign of the fluctuation, z_k, to construct its investment strategy. We have very strong indications that the sign dynamics at Δ-trading time is Markovian.

Starting from this sequence we create a Markov process approximating the symbolic sequence. In a Markov process of order m the probability to have the symbol z_n at the step n depends only on the state of the process at the previous m steps $n-1$, $n-2,\dots,n-m$.

Given a sequence of m symbols $Z_m = \{z^{(j_1)}, z^{(j_2)}, \dots, z^{(j_m)}\}$, we define $N(Z_m)$ the number of sequences Z_m and $N(Z_m, j)$ the number of times the symbol $z^{(j)}$ comes after the sequence Z_m. The transition matrix of the Markov process of order m is

$$V_{Z_m,j} \equiv \frac{N(Z_m, j)}{N(Z_m)} \tag{11}$$

and the probability of the sequence Z_m is

$$P(Z_m) \equiv \frac{N(Z_m)}{\mathcal{N}_m}\,, \tag{12}$$

where $\mathcal{N}_m = L + 1 - m$ is the total number of possible sequence of length m including superposition (L is the sequence length). It can be shown (see e.g. Ref. [18]) that the definitions (11) and (12) are coherent in the framework of ergodic Markov processes.

In the case of a Markov chain (i.e., process of order 1) the transition matrix $V_{i,j}$, i.e., the probability of a transition in one step to the state $z^{(j)}$ starting from the state $z^{(i)}$, contains all the relevant information for the process. Naming $N(i)$ the number of symbol $z^{(i)}$ and $N(i,j)$ the number of symbol $z^{(j)}$ which comes after the symbol $z^{(i)}$, the transition matrix is

$$V_{i,j} = \frac{N(i,j)}{N(i)}\,. \tag{13}$$

If the process is ergodic the probability P_i of the state $z^{(i)}$ is given by

$$P_j = \sum_{i=1}^{2} P_i V_{i,j} = \lim_{n\to\infty} (V^n)_{i,j} \quad \forall i\,,$$

where V^n is the nth power of the matrix V.

In the following we check, using various statistical approaches, if the Markovian approximation of order m mimics properly the price behavior. First of all, we perform an autocorrelation analysis, which is the most common test in financial econometrics. Then we show that the model reproduces properly the Shannon entropy. This quantity, as discovered by Kelly [19] and shown in Baviera et al. [11], has a clear financial meaning and it plays a central role in this field. Finally, we test the validity of our approximation using more sophisticated statistical tools of information theory. We compare the mutual information of the signal to the one of the approximation and we measure a "distance" between the symbolic dynamics process and the Markovian approximation, with a technique based on the Kullback entropy.

The results we show in the following analysis have been obtained choosing $\Delta = 4 \times 10^{-4}$, but they are totally independent from this choice for $\Delta > \gamma^{\mathrm{Typ}}$.

3.1. Autocorrelation function

A standard approach to verify temporal independence of processes is the measure of autocorrelation function. A first simple similarity test can be based on it: if two processes have the same autocorrelation function it means they lose memory of their past in a similar way.

Typically, one defines *short memory* series if an exponential decay of autocorrelation function occurs [20].

The autocorrelation function of a random process x_t is

$$C(n) = \frac{\langle x_{t+n} x_t \rangle - \langle x_t \rangle^2}{\langle x_t^2 \rangle - \langle x_t \rangle^2} , \tag{14}$$

where $\langle \cdot \rangle$ indicates the sequence average introduced in Eq. (7).

The autocorrelation function can be easily computed for a Markov chain described by the transition matrix $V_{i,j}$ and the probability P_i of the state $z^{(i)}$:

$$C^{(M)}(n) = \frac{\sum_{i,j} z^{(i)} z^{(j)} (V^n)_{i,j} - (\sum_i z^{(i)} P_i)^2}{\sum_i z^{(i)2} P_i - (\sum_i z^{(i)} P_i)^2} . \tag{15}$$

If the Markov chain is ergodic one has

$$C^{(M)}(n) \sim \mathrm{e}^{(\ln|\lambda_2|)n} \quad \text{for large } n , \tag{16}$$

where λ_2 is the second eigenvalue of the transaction matrix [18]. In Fig. 4, we compare the autocorrelation functions of the return $\{\rho_k\}$ and of the sequences of the same length generated by Markov processes of various orders. We show also the theoretical value for a Markov chain (see Eq. (16)). We observe there is a very good agreement between the return sequence and the Markov process inside the statistical error. This error, for the autocorrelation function measured from a finite data set, is of the order of $O(L/\tau_c)^{-1/2}$ where τ_c is the correlation time and L is the length of the sequence ($L \simeq 160\,000$ for $\Delta = 4 \times 10^{-4}$). This corresponds to the value of the "plateau" in Fig. 4.

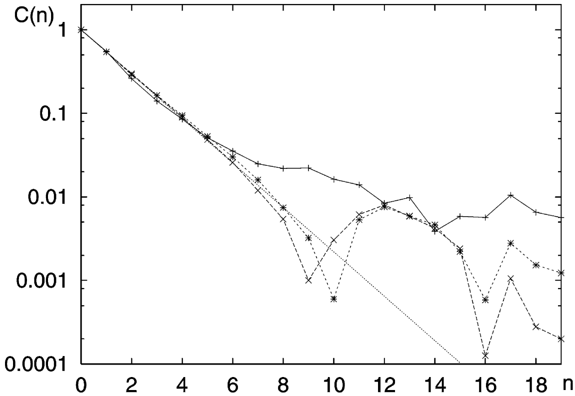

Fig. 4. Autocorrelation function of return sequence $\{\rho_k\}$ $(+)$, Markov sequence of order 1 $(\times)$ and Markov sequence of order 3$(*)$. The line is the asymptotic analytical result for a Markov process of order 1 (see Eq. (16)).

The same results are obtained if one computes autocorrelations of the symbolic return sequence $\{z_k\}$. This is a consequence of the fact that the probability density function of ρ_k is peaked around the value $\pm\Delta$.

3.2. Entropic analysis

Let us briefly recall some basic concepts of information theory and discuss the meaning of entropy in financial data analysis.

Given a symbolic sequence $Z_n = \{z^{(j_1)}, z^{(j_2)}, \ldots, z^{(j_n)}\}$ of length n with probability $p(Z_n)$, we define the block entropy H_n as

$$H_n \equiv -\sum_{Z_n} p(Z_n) \ln p(Z_n) \,. \tag{17}$$

The difference

$$h_n \equiv H_{n+1} - H_n \tag{18}$$

represents the average information needed to specify the symbol z_{n+1} given the previous knowledge of the sequence $\{z_1, z_2, \ldots, z_n\}$. The series of h_n is monotonically not increasing and for an *ergodic* process one has

$$h = \lim_{n\to\infty} h_n \,, \tag{19}$$

where h is the Shannon entropy [13].

The maximum value of h is ln(2) (this is because we are considering only two-symbols sequence). This value is reached if the process is totally uncorrelated and the symbols have the same probability. We indicate with *available* information [11] the difference between the maximum entropy and its real value:

$$I \equiv \ln(2) - h \,.$$

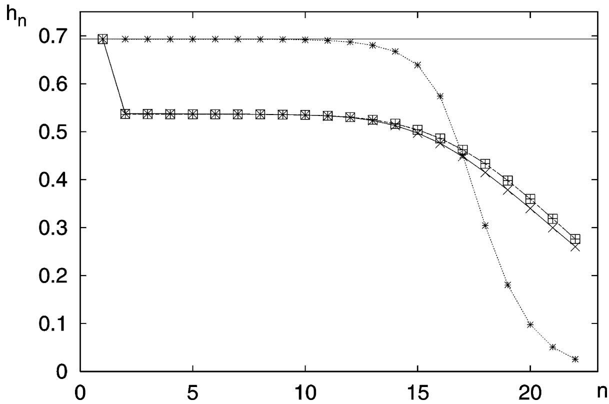

Fig. 5. h_n vs. n for the symbolic return sequence $\{z_k\}$ ($\times$), the Markov sequence of order 1 ($+$) and of order 3 ($\square$) (these entropies are almost indistinguishable). We also plot the h_n for a random walk ($*$) with its theoretical value log(2). Both the Markov process and the random walk have the same number of elements of the financial data with $\Delta = 4 \times 10^{-4}$.

If the stochastic process $\{z_1, z_2, \ldots\}$ is Markovian of order m then $h_n = h$ for $n \geqslant m$ [21]. We observe that the Markov process of order m described by Eqs. (11) and (12) has, by definition, the same entropy h_n of the symbolic sequence for all n not greater than m. In this way we can build a Markovian approximation which mimics the originary entropy as well as we desire.

From Fig. 5 one observes that the h_n are consistent with those ones of a Markov process of order 1 (the asymptotic value h is reached approximately in one step). It is therefore natural to conjecture that such a stochastic process is able to mimic price variations. Using Markov processes of order greater than one does not improve the approximation for the entropy value in an appreciable way. The value of h_n is statistically relevant till the length n of the sequence is of the same order of $(1/h)\log(L)$ [21]: this explains the folding of h_n at large n.

Why *available* information is so important in the financial context?

Kelly has shown the link between *available* information and the optimal growth rate of a capital in some particular investments [19]. A similar idea can be applied to the *patient* investor [11], i.e., a speculator who waits to modify his investment till a fluctuation of size Δ is present. He observes an *available* information different from zero. Instead, as shown in Fig. 5, for a random walk the *available* information is zero.

If the returns $\{\rho_k\}$ at fixed Δ are ruled by a Markov chain and if one neglects the transaction costs, Baviera et al. prove that the optimal growth rate of the capital is equal to the *available* information [11]. The case with transaction costs is considered in Ref. [22].

Inside a Markovian description the *available* information suggests also the order of the process one has to consider. It is useless to include the information coming from one further step in the past, if one does not observe a significant increasing of the *available* information involved in the operation.

Furthermore, if the Markovian approximation well describes the *available* information, it is not so important for a speculator who wants to maximize the growth rate of his capital that the Markovian mimicking is no longer good for other quantities. However, we shall show in the following that the Markovian approximation not only reproduces the *available* information but, in addition it is a proper description of the return dynamics itself. This is important in the case one performs a more complex investment for which is essential a detailed model for price variation (see Ref. [23]).

3.3. Mutual information

The mutual information is a measure of the average information one has about an event q knowing the result of another event s. In our case the events are the values of a process at different time.

Following Shannon we define

$$I(n) = -\iint P(x_t, x_{t+n}) \ln \frac{P(x_t, x_{t+n})}{P(x_t)P(x_{t+n})} \, \mathrm{d}x_t \, \mathrm{d}x_{t+n} \,, \tag{20}$$

where $P(x_t, x_{t+n})$ is the joint probability of the variables x_t and x_{t+n}, and $P(x_t)$ is the probability density function of the x_t.

The main advantage of this technique, compared with autocorrelation function, is that $I(n)$ is an intrinsic property of the process, i.e., it has the same value if we use x_t or a function of it. This is because $I(n)$ depends on the probability density function in such a way that the integral (20) is invariant under the change of variable $x \to y = f(x)$.

For a Markov chain the mutual information is

$$I^{(M)}(n) = -\sum_{i,j} P_i (V^n)_{i,j} \ln \frac{P_i (V^n)_{i,j}}{P_i P_j} \tag{21}$$

and, if the Markov chain is ergodic, for large n one has

$$I^{(M)}(n) \sim \mathrm{e}^{2(\ln|\lambda_2|)n} \,, \tag{22}$$

where λ_2 is again the second eigenvalue of the transaction matrix.

In Fig. 6, we compare the mutual information of the symbolic return series $\{z_k\}$ with the sequences obtained starting from Markov processes of various orders, and also with the expected theoretical value for a Markov chain (see Eq. (22)). In the case of the mutual information the statistical error is of the order L^{-1} (see also Ref. [24]) for a sequence of length L, since it is computed starting from probability distribution.

Let us stress that the good agreement between the return sequence and the Markov process for both mutual information (Fig. 6) and correlation function (Fig. 6) is a clear indicator of the validity of the Markovian approximation.

3.4. Kullback entropy

Given two discrete random variables P, Q which can assume only m different value with probabilities p_i, q_i $(i = 1, \dots, m)$, respectively, the Kullback entropy of the

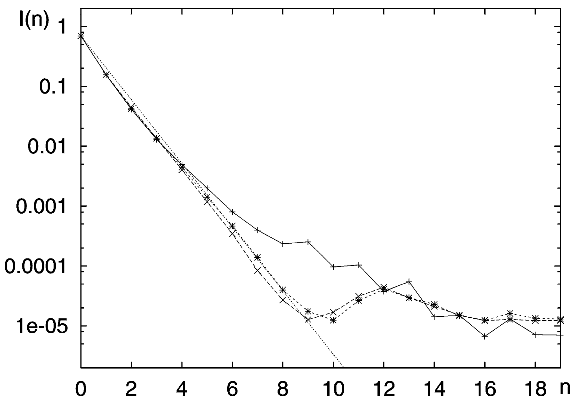

Fig. 6. Mutual information of the symbolic return sequence $\{z_k\}$ (+), Markov sequence of order 1 (×) and Markov sequence of order 3(∗). The line is the asymptotic analytical result for a Markov process of order 1 (see Eq. (22)).

probability distribution of the variable P with respect to Q:

$$J(P|Q) \equiv \sum_{i=1}^{m} p_i \ln \frac{p_i}{q_i} \tag{23}$$

is a powerful tool to measure the "distance" between the PDF of those variables. In fact, it is shown by Kullback [14] that the function $J(P|Q)$ is identically zero only if the two random variables have the same probabilities, i.e., $p_i = q_i\ \forall i$; otherwise $J(P|Q) > 0$. The $J(P|Q)$ is not a symmetric function of the two random variable. To define symmetric "distance" between PDF [14] we define the divergence between p_i, q_i as

$$K(P|Q) \equiv J(P|Q) + J(Q|P) = \sum_{i=1}^{m} (p_i - q_i) \ln \frac{p_i}{q_i} \equiv \sum_{i=1}^{m} (q_i - p_i) \ln \frac{q_i}{p_i} \,. \tag{24}$$

This function is still positive definite and it is also symmetric.

This concerns a probabilistic "distance" between random variable, but we are interested to test the similarity of two stochastic processes. To perform such a test we consider a symbolic sequence of length n, $Z_n = \{z^{(j_1)}, z^{(j_2)}, \ldots, z^{(j_n)}\}$ obtained from the processes we are interested in, and we compute the "distance" in the Kullback way for the PDF of such sequences for all n. When $n \to \infty$ we test the similarity of the processes as a whole, but, as happens for the entropy analysis, we expect for n a limit due to statistical errors.

In Fig. 7, we show $K_n(P|Q)$ vs. n where P and Q are, respectively, the Markov process and the financial process. The sequence Z_n for the financial quotes are obtained from the symbolic return sequence $\{z_k\}$, and its probability is numerically computed as

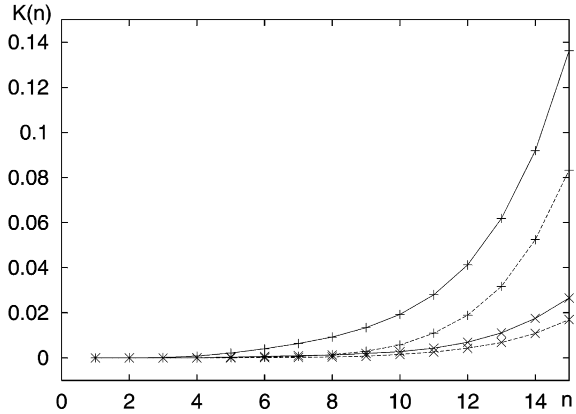

Fig. 7. $K_n(P,Q)$ vs. n. The symbols indicates the order of the Markov process involved in the similarity test: (+) indicate the first order and (×) indicate the third order. The full line and the dashed line indicate the comparison between the theoretical Markov process with the symbolic return series and the theoretical Markov process with the symbolic Markov sequence, respectively.

in Eq. (12). For the Markov process the probabilities of the Z_n are calculated starting from the transition matrix V. In the case of a Markov chain we have

$$p_n = p_{j_1} \cdot V_{j_1,j_2} \cdot V_{j_2,j_3} \cdots V_{j_{n-1},j_n} \, .$$

We have also calculated the Kullback entropies between the PDF of the Markov process computed theoretically using the transition matrix, and numerically with a symbolic sequence (of the same length of the financial sequence). This shows the relevance of the statistical error, and gives an indication of the order of n at which one must stop to have statistically relevant results.

All the Kullback tests are performed using Markov process of order 1 and 3 and is evident from Fig. 7 how this last process well reproduce the financial series.

For the Kullback entropies the statistical error increases with n. In order to have reasonable statistical we must restrict ourselves to take $n \leqslant (1/h)\log(L)$, as in the case of entropy analysis.

At the end of this section we recall that the symbolic return series $\{z_k\}$ is chosen at fixed Δ, but the results of our analysis, i.e., the good agreement with a Markov process, is strongly independent from the value of Δ.

4. Conclusions

In econometric analysis it is obviously relevant to test the validity of “random walk” models because of their strict link with market efficiency. In this paper we have first discarded “random walk” models as proper description of some features of the financial signal and then built a model considering the Deutschemark/US dollar quotes in the period from October 1, 1992 to September 30, 1993.

We have developed a technique, based on the measure of exit times PDF, which allows us to reject the random walk hypothesis. The presence of strong correlations has been observed by means of an "anomalous" scaling of $\langle\tau\rangle$ and the presence of a power-law behavior of the exit times probability density function $P_\Delta(\tau)$ for $\Delta > \gamma^{\mathrm{Typ}}$. This implies the failure of the "random walk" models where $\langle\tau\rangle$ scales as Δ^2 and the $P_\Delta(\tau)$ tail is exponential.

We want to interpret γ^{Typ} as a natural cut-off due to the absence of a profitable trading rule for a *patient* investor with Δ less than γ^{Typ}: in this case profits are less than costs. We recall that a *patient* investor cares only of the quotes where it is present a price variation at least of a percentage Δ.

The main advantage of this approach is that it shows that this class of models gives an inadequate description even at the qualitative level and it suggests a new point of view in financial analysis.

Instead of considering an arbitrary measure of time we suggest to limit the analysis only at the times when something relevant from the financial point of view happens. In particular, we focus our attention on return fluctuations of size Δ. This analysis is equivalent to the behavior of a *patient* speculator.

We show that the sign of the returns of such a sequence can be approximated by a Markovian model by means of several tests. We have considered the usual autocorrelation approach obtaining a very good agreement inside the statistical errors. In spite of its simplicity the autocorrelation analysis has the disadvantage that gives different results if one considers the random variable x_t or a function $f(x_t)$ of it. The mutual information is a generalization of this tool which does not depend from the function f considered. The agreement observed for the mutual information is surely a severe test of similarity.

A central role in the comparison between the approximation and the "true" signal is surely played by the *available* information. Following the idea of Kelly, it has been shown by Baviera et al. [11] that this quantity corresponds, in absence of transaction costs, to the optimal growth rate of the invested capital following a particular trading rule. We show that even a Markov process of order 1 mimics properly the Deutschemark/US dollar behavior.

Finally, we have analyzed a "distance" between processes based on the Kullback and Leibler entropy. The advantage of this technique is that one is able to estimate the difference between the processes with a quantity strictly connected with Shannon entropy and then with the *available* information, i.e., the quantity of interest for a *patient* speculator.

The Markovian model we have considered in this paper not only has the advantage to mimic very well the *available* information of the financial series but also give strong indication about the "true" process itself. While the former is the quantity of interest for a *patient* speculator who invests directly on the exchange market, the latter is more interesting from both the theoretical and experimental sides to have a deeper insight of the market behavior. This asset-model will allow to reach a better evaluation of risk with the natural consequences on the derivative field.

Acknowledgements

We would like to thank Michele Pasquini and Maurizio Serva for the infinite number of discussion on the subject.

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