Gigi Model:
a simple stochastic volatility approach for multifactor interest rates

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Abstract

A parsimonious model for interest rates' term structure is devised to take into account both volatility smile and multifactor dynamics. We propose a stochastic volatility generalization of the Bond Market Model that allows to price caps and floors with one-dimensional easy-to-handle closed formulas. For each caplet/floorlet the model generalizes the classical Black formula with 1 free parameter (the implied volatility) to a scheme with 3 parameters, each one responsible for one characteristic of the implied curve (average volatility, vol-of-vol, skew).

On a given set of reset dates, Monte Carlo simulations are straightforward even in the spot measure, due to the simplicity of the dynamics modeled as a Markov chain.

A comparison with an implied volatility approach is discussed. Calibration issues are described in detail and a good agreement to the EURO cap/floor market is found.

Keywords: Stochastic volatility, Cap/Floor, Multifactor interest rate model, Bond Market Model.

JEL Classification: $E43$, $G13$

1 Introduction

The success of Black models in interest rates' derivatives is probably due to the elementary analytical tractability of this closed form solution, crucial in market calibration, and the simplicity of the underlying dynamics that is helpful in implementing numerical algorithms for exotics. Unfortunately, it is common knowledge that a Black description is not adequate to describe the observed market prices of caps and floors (with strikes different from the At-The-Money) quoted by brokers in the OTC market, because one gets different *implied volatilities* for different strikes: a feature known as volatility smile.

In this paper we introduce Gigi model, a multifactor stochastic volatility model for IR term structure that, maintaining the essential characteristics of market models, is able to capture the salient features of the caplet/floorlet smile and some stylized facts in its time evolution [13]. It leads to elementary closed form formulas for plain vanillas, straightforward calibration of market model parameters and a simple underlying dynamics that allows an easy implementation of Monte Carlo algorithms when pricing exotics.

We devise a parsimonious model for the term structure of interest rates where volatility smile is properly represented. In this paper we adopt a practitioner's point of view common in exotics derivatives structuring and trading desks: given a set of reset dates and market information on plain vanilla instruments, we build a model in order to price IR exotic derivatives with the specified set of reset dates. In order to keep the presentation of the paper's idea as elementary as possible without sacrificing mathematical correctness, we avoid generality in favor of simplicity; in particular we describe the dynamics between a discrete grid of time points, this simplifies considerably the notation and the mathematics involved.

An interest rates' model to be relevant should describe accurately the set of *implied curves*, set that is improperly called *implied volatility surface*. Unfortunately, cap/floor *implied volatility surface* cannot be reconciled within an elementary displaced-diffusion approach [14, 22] as discussed in detail in [22], and then a Black-like formula is not adequate to describe properly a caplet/floorlet. Moreover, as already stressed in the literature (see e.g. [13]), it is not enough having a model that fits the surface but the model should also

- allow a simple description not only in the "natural" forward measure but also in the spot measure: a fundamental property when implementing numerical algorithms for exotics;
- be a parsimonious model;
- reproduce the observed evolution of the *implied volatility surface*;
- reflect the relevant risks associated with an *implied volatility curve* as a function of the strike.

Standard single-factor stochastic volatility models commonly used in the equity and FX markets (e.g. [15] and [18, 19]) cannot be generalized in an elementary way to the multifactor interest rate case. In the work of [11] the authors have proposed a model where forward Libor rates are driven by Levy processes; they derive an explicit formula for caplets/floorlets which uses bilateral Laplace transforms. Unfortunately "this model may be not so straightforward to calibrate, and numerical procedures for pricing exotic derivatives are usually rather difficult to implement", as pointed out by [6] on page 452.

Recently [7] have introduced the Uncertain-Volatility Models (UVM hereinafter), a class of simple stochastic volatility models that allow a direct generalization to the multifactor interest rate case [6] and simple Monte Carlo simulations.

Gigi model is a stochastic volatility multifactor IR model that shows the above properties and it preserves UVM elementary description of underlying dynamics, a feature that is essential when pricing exotics that are either path dependent [6, 4] or with callable characteristics [12, 3]. It is much simpler than [11]: it focuses on a discrete time grid with a dynamics as simple as an UVM with volatility modeled as a Levy-stable random variable. In particular Gigi model allows us to write plain vanilla European caplets/floorlets as one-dimensional closed formulas (as simple as Black formulas) similar to the Bond Market Model ones [2]: this fact allows a straightforward calibration on market data and a direct financial interpretation of model parameters. Model dynamics is elementary and Monte Carlo simulations are straightforward even in the spot measure.

An outline of the paper is as follows. In the next section we introduce the class of Bivariate Bond Market Models (BBMM hereinafter) as a generalization of UVM. The basic properties of this class of models are presented in section **3**. In section **4** we show how to get, within this class of models, easy-to-handle formulas for European caplets/floorlets (both plain vanilla and digital) that can be written in closed one-dimensional form. In section **5** Gigi model is introduced as the simplest BBMM and we show that Monte Carlo simulations are easy to be implemented even in the spot measure. In section **6** we discuss how symmetry in the *implied volatility* is reflected in model parameters. Calibration issues for Gigi model are also discussed in detail. Finally in section **7** we summarize the main results and we state some concluding remarks.

2 Bivariate Bond Market Models

In this section we introduce a broad class of stochastic volatility models, class that is possible to manage with very simple numerical and analytical tools; Gigi model, presented in section **5**, is a particular example within this class. BBMM allows us to price and manage both plain vanillas and exotics in multifactor interest rate markets in a way as simple as an Uncertain-Volatility Model.

UVM is a valid alternative to standard stochastic volatility models where volatility is described by a diffusion process; UVM approach preserves analytical tractability while showing a smile surface in the *implied volatility*. These models are the simplest stochastic description of asset's volatility: volatility is modeled by a simple random variable that can take only a finite set of values and it is independent from the Brownian motion in the underlying dynamics. UVM can be viewed as a Black model [5] where one assumes several possible scenarios for underlying's volatility. Therefore, each UVM enjoys a great analytical tractability and it very simple from a numerical point of view, since we know how to handle the pricing problems given a volatility scenario.

UVM approach assumes that, although the volatility scenario is not known at value date t_0, we know the full scenario one instant later. For most derivatives it does not matter when the information is revealed. However this is not the case of interest rate markets; for example in callable products it is essential to model the information structure: in fact, if the full future scenario of volatility is revealed at time t_{0+}, the option holder does not take into account the correct smile when pricing his right to call the derivative at a future date and in his decision whether to exercise the option or not. Having revealed the scenario at time t_{0+}, at each exercise date the holder prices his callable option contingent on a given scenario: therefore, the volatility has a deterministic value and it does not show the correct smile.

In this paper volatilities are drawn on a given set of dates; volatility information is revealed step by step immediately after each date in the set, with an approach that is reminiscent of [20]. The great advantage of BBMM is that it allows to price according to simple 1-dimensional closed formulas both digital and plain vanilla caplets/floorlets. Moreover, it is well know that some parameters in UVM are not so easy to obtain in the calibration process (e.g. the probabilities of the different volatility scenarios); we devise some models with an easy calibration procedure and with all parameters that have a clear financial interpretation.

In order to present BBMM keeping the mathematical description as simple as possible, we introduce a partition in the time interval between the value date t_0 and a finite horizon date t^* (e.g. the last payment date in the derivative of interest); furthermore, model's discrete version is better suited for practical purposes. The dates in the collection $\mathcal{T} \equiv \{\hat{t}_i\}_{i=0,..,M}$ are indexed in increasing order starting from value date t_0

$$t_0 = \hat{t}_0 < \hat{t}_1 < .. < \hat{t}_M = t^* \quad .$$

These dates should include the set of reset dates $\{t_i\}_{i=0,..,N+1}$, i.e. fixing and payment dates for the exotic instrument of interest. Additional dates can be added in order to achieve the desired resolution of the time axis; the smaller the length of these time intervals, the finer the achieved resolution of the term structure dynamics. We also assume that the lags between the dates in the collection can be represented as a multiple of a given time unit Δt (day, minute, second, etc.), i.e. we can always indicate the number of time units (days, minutes, seconds, etc.) that compose each lag.

As already stressed in the introduction, the objective of this paper is to model interest rates' dynamics in order to properly describe any interest rate exotic where each payoff is determined or paid in one of the reset dates.

As in [2] we consider the dynamics for $B_i(t)$, the forward price in t of the zero coupon bond (ZC in the following) starting in t_i which pays 1 in t_{i+1}. Forward ZC price $B_i(t)$ evolves up to the fixing date t_i, after the fixing volatility is zero and the dynamics for $B_i(t)$ is frozen. In [2] it is shown that, given this condition on the instantaneous volatility, the evolution of IR term structure between value date t_0 and t_{N+1} is completely described by the dynamics of the set of forward ZC $\{B_i(t)\}_{i=0,..,N}$; in particular $B_i(t)$ describes the term structure between t_i and t_{i+1}.

The return associated to B_i between a start date $s \in \mathcal{T}$ and an end date $t \in \mathcal{T}$ with $t > s$ is

$$x_i(s,t) \equiv \ln \frac{B_i(t)}{B_i(s)} \quad .$$

In BBMM we assume that x_i volatility between two consecutive dates in $\mathcal{T}$ is modeled by the random variable

$$\hat{\sigma}_i(l) \equiv \frac{\nu_i}{\mathcal{V}_i} v_i(\hat{t}_l, \hat{t}_{l+1}) \in \Re^d \tag{1}$$

with ν_i a constant vector in $\Re^d$, ρ a correlation matrix in $\Re^d$, $\mathcal{V}_i^2 \equiv \nu_i \cdot \rho \, \nu_i$ where $\nu_i \cdot \rho \, \nu_i$ is the scalar product in $\Re^d$ between ν_i and $\rho \, \nu_i$, and $v_i^2(\hat{t}_l, \hat{t}_{l+1})$ is a positive Levy-stable random variable drawn immediately after time $\hat{t}_l$ and defined according to

$$v_i^2(\hat{t}_l,\hat{t}_{l+1}) = \begin{cases} \displaystyle\sum_{n=1}^{(\hat{t}_{l+1}-\hat{t}_l)/\Delta t} \frac{\Delta t}{\hat{t}_{l+1}-\hat{t}_l}\hat{v}_i^2(n) & \hat{t}_l < t_i \\ 0 & otherwise \end{cases}$$

where $\hat{v}_i^2(n)$ are i.i.d. positive Levy-stable random variables with mean $\mathcal{V}_i^2$ and they are independent from all other volatilities for $j \neq i$.

The average volatility between a start date $s \in \mathcal{T}$ and an end date $t \in \mathcal{T}$ with $t > s$ (vol hereinafter), defined as

$$v_i^2(s,t) \equiv \frac{1}{t-s} \sum_{l:\, s\le\hat{t}_l<t} \left(\hat{t}_{l+1}-\hat{t}_l\right) v_i^2(\hat{t}_l,\hat{t}_{l+1}) = \frac{1}{t-s} \sum_{l:\, s\le\hat{t}_l<t} \left(\hat{t}_{l+1}-\hat{t}_l\right) \hat{\sigma}_i(l)\cdot\rho\hat{\sigma}_i(l) \ , \tag{2}$$

is also a positive Levy-stable random variable whose distribution is independent from the chosen partition $\mathcal{T}$. In this case the Laplace transform of the random variable $(t-s)\; v_i^2(s,t)$ is

$$\mathcal{L}_i[\omega; s,t] \equiv \int_0^{+\infty} dv_i^2(s,t) e^{-\omega\,(t-s)\; v_i^2(s,t)} \wp\left[v_i^2(s,t)\right] \tag{3}$$

with

$$\ln\mathcal{L}_i\left[\omega; s,t\right] = (t-s)\psi_i\left[\omega\right] \qquad \forall\; s,t\in\mathcal{T},\; t>s$$

and $\psi_i\left[\omega\right]$ is a time independent function. It is possible to show that $\psi_i\left[\omega\right]$ has values in $\Re$ when ω is in the set $\mathcal{A}\subseteq\Re$. Let us notice that the set $\mathcal{A}$ is not empty since it always includes the zero.

BBMM return dynamics in the t_i-forward measure is

$$x_i(s,t) \equiv -\frac{1+\eta_i}{2} \sum_{l:\, s\le\hat{t}_l<t} \left(\hat{t}_{l+1}-\hat{t}_l\right) \hat{\sigma}_i(l)\cdot\rho\;\hat{\sigma}_i(l) + \sum_{l:\, s\le\hat{t}_l<t} \sqrt{\hat{t}_{l+1}-\hat{t}_l}\hat{\sigma}_i(l)\cdot\xi^{(i)}(l) - \mathcal{L}_i\left[\frac{\eta_i}{2}; s,t\right] \tag{4}$$

with the parameter $\eta_i/2 \in \mathcal{A}$ and $\xi^{(i)}(l)$ a vector in $\Re^d$ of zero mean Gaussian random variables drawn immediately after time $\hat{t}_l$ s.t.

$$Corr\left(\xi_n^{(i)}(l), \xi_j^{(i)}(m)\right) = \rho_{n\,j}\;\delta_{l\,m}$$

and the $\xi^{(i)}$ are independent from volatilities. According to standard notation δ_{lm} is Kroneker's delta. Using the properties of Gaussian random variables and volatility definition (1), we can rewrite (4) as

$$x_i(s,t) = -(t-s)\left\{\frac{1+\eta_i}{2}v_i^2(s,t) + \psi_i\left[\frac{\eta_i}{2}\right]\right\} + \sqrt{t-s}\; v_i(s,t) g_i(s,t) \tag{5}$$

where $g_i(s,t)$ is a standard normal variable and $v_i^2(s,t)$ is the average volatility between s and t. A model in the BBMM class is fully specified once we define the collection of dates $\mathcal{T}$ and the probability distribution of the volatilities.

Some comparisons with the existing literature can be interesting at this point.
In the Bond Market Model [2] the return $x_i(s,t)$ is Gaussian and it can be written as

$$x_i(s,t) = -\frac{1}{2}(t-s)v_i^2(s,t) + \sqrt{t-s}\, v_i(s,t) g_i(s,t)$$

where $g_i(s,t)$ is a standard normal variable as above and $v_i^2(s,t)$ is a deterministic average volatility between s and t. BBMM parameter η_i generalizes the drift term of x_i in BMM dynamics and, as we show in section **5**, η_i controls the skew of the *implied volatility curve*. In BBMM we generalize BMM deterministic volatility to a stochastic one; we extend the BMM Gaussian return $x_i(s,t)$ to a broader class of random variables. It is then clear why these models are called Bivariate Bond Market Models: the return between s and t depends on the pair of variables $(g_i(s,t), v_i(s,t))$.
We also observe that $x_i(s,t)$ depends from the volatility only through $v_i^2(s,t)$ and, similarly to the BMM case, $x_i(s,t)$ conditioned on $v_i^2(s,t)$ is described by the Gaussian probability distribution

$$\wp\left[x_i(s,t)|v_i^2(s,t)\right] = \frac{1}{\sqrt{2\pi(t-s)v_i^2(s,t)}} \exp - \frac{\left\{x_i(s,t) + \frac{1+\eta_i}{2}(t-s)v_i^2(s,t) + \ln \mathcal{L}_i\left[\eta_i/2; s,t\right]\right\}^2}{2(t-s)v_i^2(s,t)} \quad . \tag{6}$$

It is also interesting to observe the relation between BBMM and Variance Gamma model [18, 19, 9]. Choosing the volatility Gamma distributed in a single factor BBMM, we get a discrete version of [9] as shown in detail in appendix **C**.
Finally, as we have already stressed in the introduction BBMM is reminescient of the UVM in the form described in [20] if we define a piecewise-constant instantaneous volatility structure between two consecutive dates in $\mathcal{T}$ as

$$\sigma_i(t) \equiv \hat{\sigma}_i(l) \in \Re^d \quad \forall t \in \left(\hat{t}_l, \hat{t}_{l+1}\right] \quad ,$$

and volatility is drawn immediately after time $\hat{t}_l$. In this case vol can be written

$$v_i^2(s,t) \equiv \frac{1}{t-s}\int_s^t du\, \sigma_i(u)\cdot\rho\sigma_i(u) \qquad \forall\, s,t \in \mathcal{T},\ t > s \qquad .$$

BBMM has a main advantage compared with UVM: the Levy-stable property allows us to write caplets/floorlets as a simple one-dimensional closed formula, as we show in section **4**.

We conclude this section mentioning that in BBMM (as in UVM) it is possible to write the return in a generic forward measure t_j with $j < i$ (See appendix **A** for details). This property is very important when implementing Monte Carlo simulations, as we discuss in detail in section **5** in the Gigi model case.

3 Basic BBMM Properties

In this section we start showing that BBMM class is a consistent framework to value both plain vanillas and exotics and it does not allow any arbitrage opportunity in the collection of dates $\mathcal{T}$; in the next section we show that is possible to write European options in closed form as simple one-dimensional integrals.

We start showing that, the characteristic function for $x_i(s,t)$ has a simple form. The characteristic function is

$$F_i[\omega; s, t] \equiv E_s^{(i)}[e^{i\omega\, x_i(s,t)}] = \int_{-\infty}^{\infty} dx_i(s,t) \wp\left[x_i(s,t)\right] e^{i\omega\, x_i(s,t)} \qquad \forall\, s,t \in \mathcal{T},\ t > s$$

where $E_s^{(i)}[\cdot]$ is the expectation under the i^{th} forward measure given the information at time s. In the next lemma we show that $F_i[\omega; s, t]$ is just a function of the Laplace transform $\mathcal{L}_i\left[\omega; s, t\right]$ defined in (3).

LEMMA:

In BBMM the characteristic function for $x_i(s,t)$ is

$$F_i[\omega; s, t] = \mathcal{L}_i\left[\frac{\omega^2 + i(1+\eta_i)\omega}{2}; s, t\right] \exp\left\{-i\omega \ln \mathcal{L}_i\left[\frac{\eta_i}{2}; s, t\right]\right\} \qquad \forall\, s,t \in \mathcal{T},\ t > s \qquad .$$

Proof: By definition the characteristic function is

$$F_i[\omega; s, t] = \int_0^{+\infty} dv_i^2(s,t) \wp\left[v_i^2(s,t)\right] \int_{-\infty}^{+\infty} dx_i(s,t) \wp\left[x_i(s,t)|v_i^2(s,t)\right] e^{i\omega\, x_i(s,t)}$$

where the conditional probability is given by equation (6) and defining

$$x_i'(s,t) = x_i(s,t) + \ln \mathcal{L}_i\left[\frac{\eta_i}{2}; s, t\right]$$

we can rewrite

$$\begin{aligned}\wp\left[x_i(s,t)|v_i^2(s,t)\right] e^{i\omega\, x_i(s,t)} \quad &= \quad \frac{1}{\sqrt{2\pi(t-s)v_i^2(s,t)}} \exp -\frac{\left[x_i'(s,t) - (t-s)\left(i\omega - (\eta_i+1)/2\right) v_i^2(s,t)\right]^2}{2(t-s)v_i^2(s,t)} \\ & \exp\left\{-i\omega \ln \mathcal{L}_i\left[\frac{\eta_i}{2}; s, t\right] - \frac{t-s}{2}\left[\omega^2 + i\omega(\eta_i+1)\right]\ v_i^2(s,t)\right\}\end{aligned}$$

The lemma is proven after observing that the integral in x_i of the first part of the above expression is equal to 1. $\diamondsuit$

The above lemma states that the characteristic function of $x_i(s,t)$, which is function of the random variables' pair $(g_i(s,t), v_i(s,t))$, can be viewed as a simple expression of the Laplace transform of (uniquely) the vol. Using the above lemma, it is also immediate to show that the basic martingale property holds.

COROLLARY:

In BBMM $B_i(t)$ is a martingale process in the t_i-forward measure, i.e.

$$E_s^{(i)}[B_i(t)] = B_i(s) \qquad \forall\, s,t \in \mathcal{T},\ t > s \qquad .$$

Proof: It is enough to show that $E_s^{(i)}[\exp[x_i(s,t)]] = 1$, that is an application of the lemma for $\omega = -i$, since $F_i[-i; s, t] = 1$. $\diamondsuit$

We have then shown that there are no arbitrage opportunities when trading B_i on the dates in the set $\mathcal{T}$.

The dynamics is similar when considering forward start ZC bond with longer tenor. With empty sums denoting zero and empty products denoting 1, let us define the forward ZC bond which starts in t_α and pays 1 in t_ω

$$B_{\alpha\omega}(t) \equiv \begin{cases} \prod_{i=\alpha}^{\omega-1} B_i(t) & if \;\; t \le t_\alpha \\ B_{\alpha\omega}(t_\alpha) & otherwise \end{cases}$$

and the volatility

$$v_{\alpha\omega}^2(s,t) \equiv \frac{1}{t-s} \sum_{i,n=\alpha}^{\omega-1} \int_s^t du\, \sigma_i(u) \cdot \rho \sigma_n(u) \qquad .$$

LEMMA:

In the t_α-forward measure the dynamics for $B_{\alpha\omega}(t)$, for $t > s$, is

$$\ln \frac{B_{\alpha\omega}(t)}{B_{\alpha\omega}(s)} = \sqrt{t-s}\, v_{\alpha\omega}(s,t) g_{\alpha\omega}(s,t) - \frac{t-s}{2} v_{\alpha\omega}^2(s,t) - (t-s) \sum_{i=\alpha}^{\omega-1} \left\{ \frac{\eta_i}{2} v_i^2(s,t) + \psi_i \left[\frac{\eta_i}{2} \right] \right\} \qquad \forall\, s,t \in \mathcal{T}. \tag{7}$$

where $g_{\alpha\omega}(s,t)$ is a standard normal variable.

Proof: Straightforward using the above definitions and equations (4) and (21) in the t_α-forward measure.$\diamondsuit$

PROPOSITION:

In BBMM $B_{\alpha\omega}(t)$ is martingale in the t_α-forward measure, i.e.

$$E_s^{(\alpha)}[B_{\alpha\omega}(t)] = B_{\alpha\omega}(s) \qquad \forall\, s,t \in \mathcal{T},\ t > s \qquad .$$

Proof: A consequence of the above lemma. $\diamondsuit$

Let us comment the above proposition which is the main result of this section. We have shown that BBMM is well defined since there is no possibility of arbitrage at any time $s \in \mathcal{T}$ trading any bond with reset dates in the set $\{t_i\}_{i=0,..,N+1}$; choosing any model within this framework our objective is accomplished for the interest rates' derivatives we are interested in. As we have already stressed in the introduction we desire to manage interest rate derivatives (exotics and plain vanillas) with reset dates in the collection $\{t_i\}_{i=0,..,N+1}$. BBMM achieves this task.

One of the main advantages of BBMM is that we are always able to reduce the solution of European options in presence of stochastic volatility to a closed one-dimensional formula as we show in the next section, there we show that it is straightforward to obtain caplets and floorlets as simple 1-dimensional Fourier transform.

4 European Option Valuation

In this section we show that European plain vanilla and digital options with maturity t_i and underlying B_i can be written in a simple closed form.

It is convenient to introduce a more compact notation: we omit the time dependence of the quantities when the start date s is the value date t_0 and the end date t is the fixing date t_i i.e.

$$\begin{array}{rcl} x_i & \equiv & x_i(t_0, t_i) \\ v_i & \equiv & v_i(t_0, t_i) \\ \mathcal{L}_i[\omega] & \equiv & \mathcal{L}_i[\omega; t_0, t_i] \ . \end{array}$$

Since, it is well known how to write option prices in terms of the characteristic function (e.g. [15, 8]), and the characteristic function is an elementary expression of the Laplace transform $\mathcal{L}_i$, we expect to find a simple closed form formula for plain vanilla option prices; in the next proposition we state this result.

The payoff $h_O(t_i)$ in t_i of the option O in the call case is

$$h_O(t_i) = \begin{cases} [B_i(t_i) - K]^+ & = & H(x_i + k) B_i(t_0)(e^{x_i} - e^{-k}) \quad \textit{plain call } P_C \\ H(B_i(t_i) - K) & = & H(x_i + k) \quad \textit{digital call } D_C \end{cases}$$

where $k \equiv \ln B_i(t_0)/K$, $H(\cdot)$ is the Heaviside step function and the digital call has unitary payoff. In the put case reads

$$h_O(t_i) = \begin{cases} [K - B_i(t_i)]^+ & = & H(-k - x_i) B_i(t_0)(e^{-k} - e^{x_i}) \quad \textit{plain put } P_P \\ H(K - B_i(t_i)) & = & H(-k - x_i) \quad \textit{digital put } D_P \end{cases} \ .$$

The price of option O at value date t_0 is

$$O = B_{0i}(t_0) E^{(i)} \left[h_O(t_i) \right] \qquad .$$

Let us define in the t_i-forward measure the following integrals that appear in the European option solution

$$\begin{array}{rcl} I_i(k) & \equiv & \displaystyle\int_{-\infty}^{+\infty} dx_i \wp \left[x_i \right] H(-k - x_i) \\ J_i(k) & \equiv & \displaystyle\int_{-\infty}^{+\infty} dx_i \wp \left[x_i \right] e^{x_i} H(-k - x_i) \ . \end{array}$$

LEMMA:

In BBMM the integrals $I_i(k)$ and $J_i(k)$ are:

$$
\begin{aligned}
I_i(k) &= H\left(\eta_i+1\right)+\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{e^{-k'[i\omega-(1+\eta_i)/2]}}{i\omega-(1+\eta_i)/2}\,\mathcal{L}_i\left[\frac{\omega^2+(1+\eta_i)^2/4}{2}\right]\\
J_i(k) &= H\left(\eta_i-1\right)+\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{e^{-k'[i\omega+(1-\eta_i)/2]}}{i\omega+(1-\eta_i)/2}\,\mathcal{L}_i\left[\frac{\omega^2+(1+\eta_i)^2/4}{2}\right]/\mathcal{L}_i\left[\frac{\eta_i}{2}\right]
\end{aligned}
\tag{8}
$$

where

$$
k' = k - \ln\mathcal{L}_i\left[\frac{\eta_i}{2}\right]
$$

and $H(\cdot)$ is the Heaviside step function and $\mathcal{L}_i\,[\omega]$ is the Laplace transform of the vol between the value date t_0 and the fixing date t_i (3).

Proof: See appendix **A**. $\diamondsuit$

PROPOSITION:

In BBMM plain vanilla ZC call option and digital ZC call options with maturity t_i and underlying B_i are

$$
\begin{aligned}
P_C(k) &= B_{0i+1}(t_0)e^{-k}(I_i(k)-e^k J_i(k)+e^k-1)\\
D_C(k) &= B_{0i}(t_0)(1-I_i(k))
\end{aligned}
$$

and plain vanilla ZC put option and digital ZC put options are

$$
\begin{aligned}
P_P(k) &= B_{0i+1}(t_0)e^{-k}(I_i(k)-e^k J_i(k))\\
D_P(k) &= B_{0i}(t_0)I_i(k) \qquad .
\end{aligned}
$$

Proof: Just an application of the above lemma. $\diamondsuit$

The above proposition can be equivalently written for caps/floors using the relation between forward Libor rates and forward ZC. There is an equivalence to consider $B_i(t)$ and the forward Libor rate at time t $L_i(t)$ of the Libor rate $L_i(t_i)$ between t_i and t_{i+1}, fixing in t_i, payment in t_{i+1} and calculated for the lag $\theta_i \equiv t_{i+1}-t_i$. They are equivalent since the following relation holds (see e.g. [22, 6]):

$$
L_i(t) = \frac{1}{\theta_i}\left(\frac{1}{B_i(t)}-1\right) \quad .
$$

The payoff of the i^{th} caplet is established at time t_i as the difference, if positive, between the Libor rate with fixing at time t_i and a strike $\mathcal{K}$. As usual, the i^{th} payoff is established in t_i, calculated for the lag θ_i and paid in t_{i+1}. In this case we have

$$k \equiv \ln \frac{1+\theta_i \mathcal{K}}{1+\theta_i L_i(t_0)} \qquad .$$

COROLLARY:

In BBMM plain vanilla and digital caplets are:

$$\begin{array}{rclcl} c_i(k) & = & \theta_i B_{0i}(t_0) E^{(i)} \left[B_i(t_i)\, (L_i(t_i) - \mathcal{K})^+ \right] & = & B_{0i}(t_0) \left(I_i(k) - e^k J_i(k) \right) \\ dc_i(k) & = & \theta_i B_{0i}(t_0) E^{(i)} \left[B_i(t_i)\, H(L_i(t_i) - \mathcal{K}) \right] & = & \theta_i\; B_{0i+1}(t_0)\; J_i(k) \end{array} \tag{9}$$

with $I_i(k)$ and $J_i(k)$ in equations (8) and

$$k' \equiv \ln \frac{1+\theta_i \mathcal{K}}{1+\theta_i L_i(t_0)} - \ln \mathcal{L}_i \left[\frac{\eta_i}{2} \right] = k - \ln \mathcal{L}_i \left[\frac{\eta_i}{2} \right] \qquad .$$

Plain vanilla and digital floorlets are

$$\begin{array}{rclcl} f_i(k) & = & \theta_i B_{0i}(t_0) E^{(i)} \left[B_i(t_i)\, (\mathcal{K} - L_i(t_i))^+ \right] & = & B_{0i}(t_0) \left(I_i(k) - e^k J_i(k) + e^k - 1 \right) \\ df_i(k) & = & \theta_i B_{0i}(t_0) E^{(i)} \left[B_i(t_i)\, H(\mathcal{K} - L_i(t_i)) \right] & = & \theta_i\; B_{0i+1}(t_0)\; (1 - J_i(k)) \end{array} \quad . \tag{10}$$

Proof: The result follows form the above proposition, after observing that, the caplet (floorlet) in the t_i-forward measure is equivalent to a put (call) option on B_i, i.e. in the caplet case

$$c_i(k) = B_{0i}(t_0)\,(1+\theta_i \mathcal{K})\, E^{(i)} \left[\frac{1}{1+\theta_i \mathcal{K}} - B_i(t_i) \right]^+ \quad . \;\diamondsuit$$

Equations (9) and (10) are the main result of this section. We have shown that within BBMM class caplets and floorlets can be priced via a one-dimensional closed formula very similar to the Black-like formula in the BMM case.

In fact, let us recall that in the BMM model (i.e. under the hypothesis of constant vol v_i) caplets are [2]

$$\begin{array}{rcl} c_{i,B}(k) & = & B_{0i}(t_0) \left\{ N\left[d_1(k; v_i^2) \right] - e^k N \left[d_2(k; v_i^2) \right] \right\} \\ dc_{i,B}(k) & = & \theta_i\; B_{0i+1}(t_0)\; N \left[d_2(k; v_i^2) \right] \end{array}$$

where

$$\left\{\begin{array}{lcl} d_1(k;v_i^2) & = & -\dfrac{k}{\sqrt{(t_i-t_0)v_i^2}}+\dfrac{1}{2}\sqrt{(t_i-t_0)v_i^2} \\ d_2(k;v_i^2) & = & -\dfrac{k}{\sqrt{(t_i-t_0)v_i^2}}-\dfrac{1}{2}\sqrt{(t_i-t_0)v_i^2} \end{array}\right. . \tag{11}$$

We then notice that in BBMM caplet solution (9) is equal to the Black like solutions in the BMM case with the substitutions of the cumulated Gaussian distributions $N[\cdot]$ with the more general cumulated distributions I_i and J_i

$$\left\{\begin{array}{lcl} N\left[d_1(k;v_i^2)\right] & -> & I_i(k) \\ N\left[d_2(k;v_i^2)\right] & -> & J_i(k) \end{array}\right. . \tag{12}$$

Moreover, as a matter of curiosity, it is interesting to point out that, after some algebra, it is possible to write in a similar way to [16], the cumulated distributions I_i and J_i as a weighted sum of cumulated Gaussian distributions

$$\begin{array}{lcl} I_i(k) & = & \displaystyle\int_0^{+\infty} dv_i^2 \wp\left[v_i^2\right] N\left[\tilde{d}_1(k;v_i^2)\right] \\ J_i(k) & = & \displaystyle\int_0^{+\infty} dv_i^2 \tilde{\wp}\left[v_i^2\right] N\left[\tilde{d}_2(k;v_i^2)\right] \end{array}$$

where

$$\left\{\begin{array}{lcl} \tilde{d}_1(k;v_i^2) & = & -\dfrac{k'}{\sqrt{(t_i-t_0)v_i^2}}+\dfrac{\eta+1}{2}\sqrt{(t_i-t_0)v_i^2} \\ \tilde{d}_2(k;v_i^2) & = & -\dfrac{k'}{\sqrt{(t_i-t_0)v_i^2}}+\dfrac{\eta-1}{2}\sqrt{(t_i-t_0)v_i^2} \end{array}\right.$$

and

$$\tilde{\wp}\left[v_i^2\right]=\frac{\wp\left[v_i^2\right]\exp\left[-\frac{\eta_i}{2}(t_i-t_0)v_i^2\right]}{\mathcal{L}_i[\eta_i/2]} .$$

Furthermore, we just mention that a consequence of the above corollary is that we can easily (and in a very fast way) obtain prices for all strikes at once through Fast Fourier Transforms as in [8], once the Laplace transform $\mathcal{L}_i$ is specified and whatever is the partition $\mathcal{T}$.

Finally, it is also useful to observe that caplets and floorlets of equations (9) and (10) are functions of the strike $\mathcal{K}$ only through k. As we stress in section **6** this fact has important consequences on the dynamics of the implied volatility curve.

In the next session we show how these results apply to Gigi model, the simplest specification within the BBMM class. In appendix **C** we mention another example of BBMM.

5 Gigi model

In the Bond Market Model is considered the particular case where volatility is a deterministic function of time. In that case each forward ZC bond B_i is lognormal. We want to generalize here that approach trying to maintain the main properties of BMM. We focus on the dynamics between reset dates; it is then useful to introduce a more compact notation

$$[\cdot]_i(l) \equiv [\cdot]_i(t_l, t_{l+1}) \qquad .$$

The properties of BMM we would like to find in a more general setting are

1. *Markov chain Property*: the forward ZC transition probability density between reset dates should be written explicitly (and easy to be implemented numerically) and this property should hold also in the spot measure in order to keep simple Monte Carlo simulations;

2. *Calibration Property*: IR plain vanilla options should have a one-dimensional closed form and model parameters should be easy to be calibrated;

3. *Parsimony Property*: few parameters should synthesize the main characteristics of model dynamics and they should allow a direct financial interpretation.

Gigi model generalizes the Bond Market Model satisfying the three above properties. We assume that the set $\mathcal{T}$ is limited to the collection of reset dates $\{t_i\}_{i=0,..,N+1}$ and the BBMM instantaneous volatility (1) is

$$\sigma_i(t) = \nu_i \sqrt{G_i(l)} \in \Re^d \;\; \forall t \in (t_l, t_{l+1}] \qquad l < i \tag{13}$$

with the random variable $G_i(l)$ an unitary mean Inverse Gaussian (IG) variable with variance $\kappa_i(l) \equiv \kappa_i/(t_{l+1} - t_l)$ and each $G_i(l)$ is independent from the other $G_l(m)$; on the discrete grid of reset dates Gigi can be also viewed as a simple multifactor generalization of a NIG process [1].

Gigi dynamics for the forward ZC bond B_i, under the t_j-forward measure ($j \le i$), is given by

$$\ln \frac{B_i(t_{l+1})}{B_i(t_l)} = \begin{cases} -(t_{l+1}-t_l)\left\{ v_i(l) \displaystyle\sum_{n=l+1\wedge j}^{i-1} R_{in} v_n(l) + \frac{1+\eta_i}{2} v_i^2(l) + \psi_i\left[\frac{\eta_i}{2}\right] \right\} + \sqrt{t_{l+1}-t_l}\, v_i(l) g_i(l) & j < i \\ -(t_{l+1}-t_l)\left\{ \dfrac{1+\eta_i}{2} v_i^2(l) + \psi_i\left[\dfrac{\eta_i}{2}\right] \right\} + \sqrt{t_{l+1}-t_l}\, v_i(l) g_i(l) & j = i \end{cases} \tag{14}$$

where the parameter $\eta_i/2 \in \mathcal{A} = (-1/(2\mathcal{V}_i^2\kappa_i), +\infty)$ and the vol between the reset dates t_l and t_{l+1} is

$$v_i^2(l) = \begin{cases} \mathcal{V}_i^2 G_i(l) & l < i \\ 0 & l \geq i \end{cases} \tag{15}$$

with $R_{in} \equiv (\nu_i \rho \nu_n)/(\mathcal{V}_i \mathcal{V}_n)$ and $\mathcal{V}_i$ the average volatility. The standard normal variables g_i are such that

$$Corr(g_i(l), g_n(m)) = \delta_{lm} R_{in}$$

and the variables $g_i(l)$ and $G_n(m)$ are independent $\forall i, l, n, m$. Equation (14) is a direct consequence of BBMM dynamics (5) and (21) in Gigi model case (13).

From (15) and the Levy-stable properties of $G_i(l)$ we get that the vol (2) between value date t_0 and fixing date t_i is

$$v_i^2 = \sum_{l=0}^{i-1} \frac{t_{l+1} - t_l}{t_i - t_0} v_i^2(l) = \mathcal{V}_i^2 G_i$$

with G_i an unitary mean IG variable with variance $\kappa_i/(t_i - t_0)$. Then we obtain that

$$\ln \mathcal{L}_i[\omega] = \frac{t_i - t_0}{\kappa_i} \left[1 - \sqrt{1 + 2\mathcal{V}_i^2 \kappa_i \omega} \right] \tag{16}$$

and

$$\psi_i[\omega] = \frac{1}{\kappa_i} \left[1 - \sqrt{1 + 2\mathcal{V}_i^2 \kappa_i \omega} \right] \quad . \tag{17}$$

Equations (14) and (17) model B_i dynamics between reset dates t_l and t_{l+1}; under the t_i-forward measure the dynamics is described by the pair of independent random variables $G_i(l)$ and $g_i(l)$: for this reason the model is called $G_i g_i$. This is the simplest generalization of the BMM with stochastic volatility.

We mention that in multifactor interest rate models Monte Carlo evaluation of exotics is often an obliged solution. Gigi model allows elementary Monte Carlo simulations not only in the "natural" forward measure but also in the spot measure. We have described the dynamics between reset dates in terms of a Markov chain: a crucial property implementing Monte Carlo simulations, especially when the payoff is either path dependent [6] or presents callable features [12, 3].

In Gigi model Monte Carlo simulation of $\{B_i\}_{i=1,..,N}$ (given the dynamics (14) in the forward measure of interest) is straightforward following the 2 steps described in appendix **D**. It is probably useful to recall that simulating an IG random variable is very simple since it involves only one uniform random variable and one χ_1^2 variable: in appendix **D** we recall the numerical properties of IG variables and we show how simple is implementing $G_i g_i$ dynamics.

In Gigi model we introduce for each expiry t_i three parameters: the average volatility $\mathcal{V}_i$, the vol-of-vol κ_i and the parameter η_i, that (as we show in the next section) is responsible for the skew. There is a straight correspondence between model parameters and financial quantities, avoiding every redundancy issue and leading to a parsimonious description of a multifactor IR model with stochastic volatility. We just mention that the number of parameters can be quite large in canonical stochastic volatility multifactor HJM model, where we allow each underlying to be correlated with the instantaneous volatility.

An example can clarify this point. Let us suppose that both the underlying dynamics and the instantaneous volatility are N dimensional: each σ_i is not only correlated with the corresponding underlying but, due to the fact that different underlyings are correlated, σ_i is correlated with all other σ_j and all other underlyings. This introduces a correlation matrix with $2N \cdot (2N-1)/2$ parameters; most of them have not a straightforward financial interpretation: the correlation between the j^{th} underlying and the hidden variable σ_i (with $j \neq i$) has no evident financial meaning.

Not only the number of parameters of the corresponding Gigi model is much lower, but also each model parameter is related to a feature of an implied volatility curve. Moreover, as we shall show in the next section, even if instantaneous volatilities are independent from the Brownian motions in underlying dynamics (or equivalently the Gaussian variables ξ in the discrete time case), implied volatilities and Libor rates are correlated, as observed in the market.

Last but not least, in Gigi model, as in every model in the BBMM class, plain vanilla caplets/floorlets can be expressed in the simple one-dimensional closed form (9) and (10), with $\mathcal{L}_i$ as in (16). This fact allows a fast calibration as we show in the next section.

6 Implied Volatility Approach and Calibration

Option prices in plain vanilla markets are often quoted by stating the implied volatility as the value of volatility which yields Black's price equal to market price. In practice, for a given maturity t_i, options with different strikes require different implied volatilities: this leads to an implied volatility curve for each t_i. Said differently, Black hypothesis on returns is not satisfied, since implied volatility curves exhibit smile and skew and in particular tail distributions are fatter than the Gaussian ones. A common wisdom among market participants is that option's implied volatility depends only on its moneyness k, a property known as *Sticky Delta* rule [10]. The *Sticky Delta* rule models the intuition that the current level of At-The-Money volatility (by far the most liquid) should remain unchanged as forward rates move. A direct consequence of *Sticky Delta* rule is on the dynamics of the implied volatility curve: Libor rates and market smiles move in the the same direction as already underlined by [13]. So, in the BBMM framework, even if we have volatilities $\{\sigma_i\}$ independent from the Gaussian variables $\{\xi_i\}$ responsible for underlying's dynamics, Libor rates and implied volatilities are correlated due to the fact that they depend from the strike only through the moneyness k.
In order to show that this "market belief" finds a correspondence on market data, in figure 1 we show the implied volatility curves as a function of the strike $\mathcal{K}$ for the yearly cap with 2 years expiry ($t_i = 2$) in the EURO market. We consider the implied volatility curve on the 8^{th} December 2005, six months before (on the 8^{th} June 2005) and six months later (on the 8^{th} June 2006). The volatilities are mid-market values registered at 11:15 a.m. C.E.T.

In figure 2 we show that *Sticky Delta* rule is a good approximation by plotting implied volatility as a function of k. It is useful to underline that the three curves are quite similar even if they correspond to very different market situations: the 8^{th} June 2005 was before the European Central Bank started increasing rates (with the market pricing the possibility of further cuts), 6 months later it was immediately after the first quarter point ECB increase (meeting of the 1^{st} December 2005) and the last curve was during the tightening cycle that has followed.
As shown in Figure 2, a *Sticky Delta* rule captures reasonably the realty even considering so different

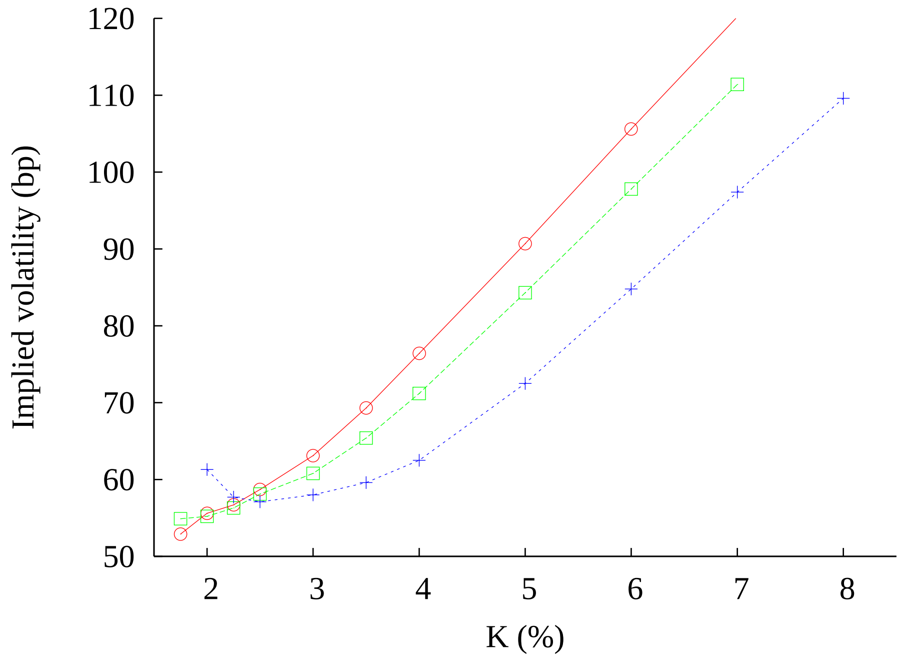

Figure 1: Implied volatilities as a function of the strike $\mathcal{K}$ for the 2y expiry yearly cap in the EURO market with value date the 8^{th} June 2005 (circles), the 8^{th} Dec 2005 (squares) and the 8^{th} June 2006 (crosses).

market situations. We recall that BBMM caplets/floorlets prices satisfy the *Sticky Delta* rule.

In this section, first, we evidence how model parameters are related to the particular shape of the smile curve, in particular that η_i controls the skew in the implied volatility curve. Second, given an implied volatility curve from market data we show how to calibrate Gigi model.

In general we obtain the corresponding implied volatility, given a set of prices for caplets/floorlets (either from the market or obtained from a model), by imposing

$$c_{i,B}(k) = c_i(k) \quad , \tag{18}$$

where $c_{i,B}(k)$ is caplet Black price.

In the *implied volatility* approach, for each k, option price is the Black formula, or stated differently we model the return as if it is Gaussian with volatility $\sigma_B(k)$. We can then write caplets (floorlets) in terms of equation (9) and (10) with

$$\ln \mathcal{L}_i[\omega] = -\omega(t_i - t_0)\sigma_B(k)^2 \qquad .$$

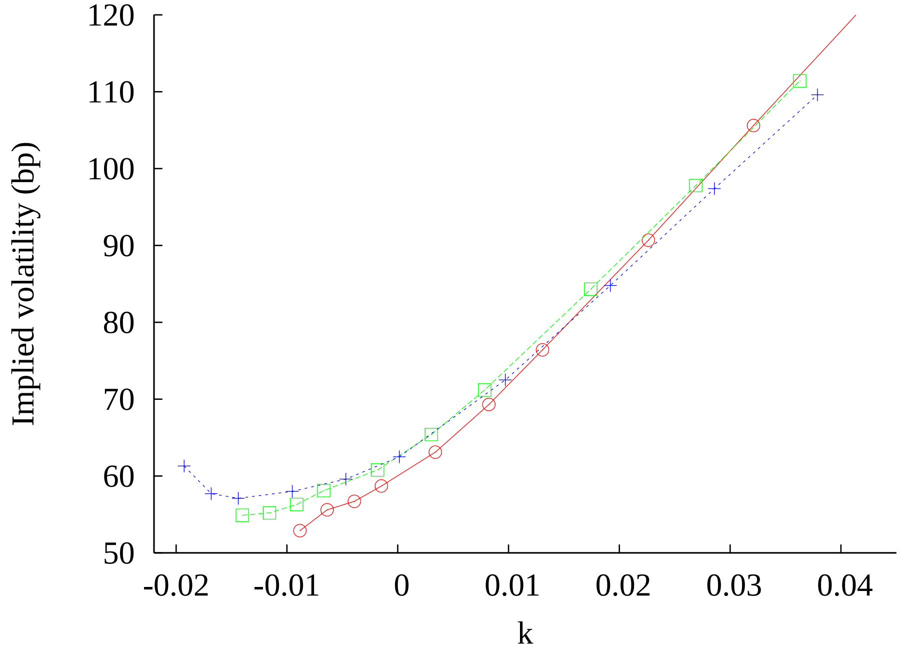

Figure 2: Implied volatilities as a function of k for the 2y expiry yearly cap of figure 1. We observe that the different curves become quite similar, the *Sticky Delta* rule appears to be quite well satisfied even in such extremely different cases.

In general the *implied volatility* is obtained inverting numerically (18), however some analytical properties of the *implied volatility* can be deduced. For example it can be useful to mention that the asymptotic behavior of $\sigma_B(k)$ for very large absolute values of k is straightforward to get as in [17], using the simple characteristic function in the Gigi case (see the lemma in section **2**).

In the following proposition we show that η_i controls the symmetry of the *implied volatility*.

PROPOSITION:

In BBMM framework, if $\eta_i = 0$ *implied volatility* curve as a function of k is symmetric, i.e.

$$\sigma_B(k) = \sigma_B(-k) \quad .$$

Proof: It is enough to prove the proposition in the caplet case. If $\eta_i = 0$ we can write the i^{th} caplet solution in BBMM framework (9)

$$c_i(k) = B_{0i}(t_0) \left\{ 1 - e^{k/2} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-ik\omega} \frac{\mathcal{L}_i \left[\left(\omega^2 + 1/4\right)/2 \right]}{\omega^2 + 1/4} \right\} \qquad .$$

Let us notice that the function of k that multiplies $e^{k/2}$ is symmetric in k, whatever is the Laplace transform $\mathcal{L}_i$.

Since Black solution in the BBMM language is

$$c_{i,B}(k) = B_{0i}(t_0)\left\{1 - e^{k/2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-ik\omega}\frac{\exp\left[-\sigma_B^2(k)(t_i - t_0)\left(\omega^2 + 1/4\right)/2\right]}{\omega^2 + 1/4}\right\}$$

and the *implied volatility* is obtained imposing (18), or equivalently

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-ik\omega}\frac{\mathcal{L}_i\left[\left(\omega^2 + 1/4\right)/2\right]}{\omega^2 + 1/4} = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{-ik\omega}\frac{\exp\left[-\sigma_B^2(k)(t_i - t_0)\left(\omega^2 + 1/4\right)/2\right]}{\omega^2 + 1/4} \quad .$$

Due to the symmetry in k of the left part of the solution the above equality is satisfied only if even the right part has the same symmetry and then the (positive) *implied volatility* is symmetric. $\diamondsuit$

IR model calibration can be either *global* or *local*. The former is a very ambitious problem and therefore is typically solved in (properly weighted) least square sense: the aim is to obtain model parameters that, on average, are able to capture underlying dynamics across all strikes; unfortunately, the price to pay is the inaccuracy in pricing individual instruments.

The latter approach, instead, can reprice exactly a small set of instruments, the ones for which the exotic we desire to manage is particularly sensitive.

As already stressed in the introduction our objective is not obtaining the same model parameterization in order to properly describe the whole plain vanilla market, but, given the (liquid) market of plain vanillas, being able of pricing and managing exotics where there is a relevant impact of the smile. We adopt here a *local* calibration that is able to capture exactly the dynamics of the hedging vehicles of interest for a given pricing problem.

In the implied volatility approach, it can be shown that a digital caplet can be priced according to the following proposition.

PROPOSITION:

In an implied volatility approach the digital caplet with expiry in t_i is

$$dc_i(k) = \theta_i\ B_{0i+1}(t_0)\ \left\{N\left[d_2(k;\sigma_B^2(k))\right] - \frac{\partial\sigma_B(k)}{\partial k}\sqrt{\frac{t_i - t_0}{2\pi}}\exp\left(-\frac{d_2(k;\sigma_B^2(k))^2}{2}\right)\right\} \qquad .$$

with $k \equiv \ln(1+\theta_i \mathcal{K})/(1+\theta_i L_i(t_0))$ and $d_2(k; \sigma_B^2(k))$ has been defined in (11).

Proof: It is well known that a digital caplet can be viewed as a caplet spread with very near strikes, i.e.

$$dc_i(k) = \lim_{\epsilon \to 0} \frac{c_{i,B}(\mathcal{K}) - c_{i,B}(\mathcal{K}+\epsilon)}{\epsilon}$$

where $c_{i,B}$ is a function of the strike also through the implied volatility $\sigma_B(k)$. Then

$$dc_i(k) = \frac{\partial k}{\partial \mathcal{K}} \left\{ -\frac{\partial c_{i,B}(k)}{\partial k} - \frac{\partial \sigma_B(k)}{\partial k} \frac{\partial c_{i,B}(k)}{\partial \sigma_B(k)} \right\} \qquad ,$$

and after simple algebra the proof follows. $\diamondsuit$

The above proposition states that, in presence of smile, the digital caplet does not only depend on the absolute level of the implied volatility $\sigma_B(k)$ but also it is sensitive to the *local* slope. The cumulative probability of having the underlying greater than the strike $\mathcal{K}$ is equal to the standard Gaussian term corrected for the presence of a non null slope in the implied volatility.

In this paper we calibrate the three parameters (average vol $\mathcal{V}_i$, vol-of-vol κ_i and skew η_i) with 3 instruments: the plain vanilla and the digital caplet on the relevant strike for the exotic of interest and a second plain vanilla on a different strike. For example in figure 3 we show the implied curve on the 8^{th} June 2006 and the one obtained in the calibration; we have supposed that the exotic derivative we are interest in is sensitive to the implied volatility near the 5% strike. The choice of the other strike is quite arbitrary; in figure 3 the second strike is equal to 2.5%. Applying a standard 3-dimensional Newton-Raphson method, the values we get for Gigi parameters are: $\mathcal{V}_i = 5.9472 \cdot 10^{-3}$, $\kappa_i = 1.077$ and $\eta_i = 151.43$.

Local calibration is extremely powerful when managing interest rate exotics that on a given strike depend not only on the level of the implied volatility but also on the local slope of the implied volatility. For all payoffs where the digital risk is remarkable (e.g. callable range accruals and autocallable products with Libor related payoffs) a *local* calibration as the one proposed here can be an adequate solution.

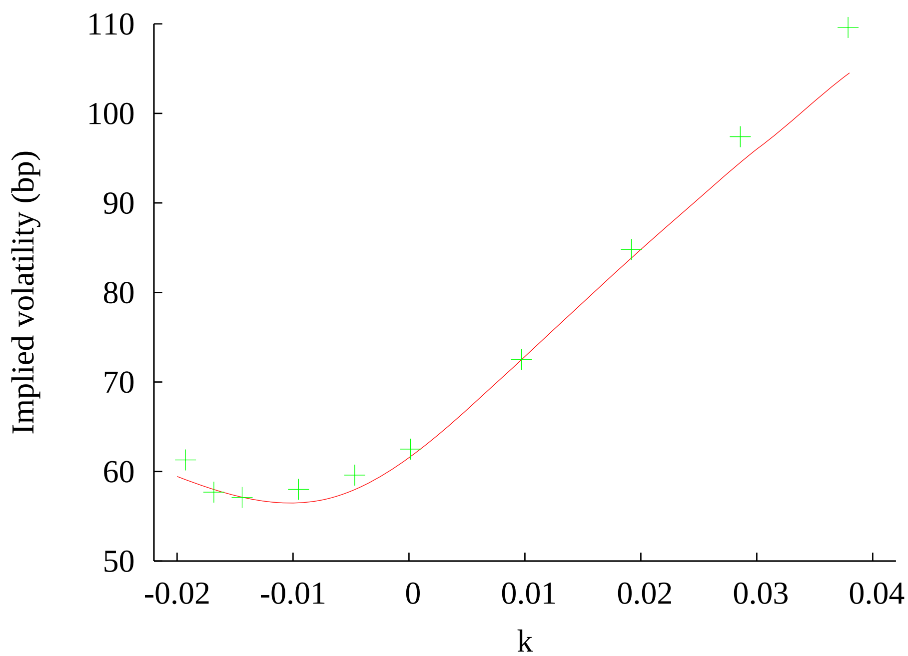

Figure 3: We plot implied volatilities as a function of k for the 2y expiry yearly cap in the EURO market with value date on the 8^{th} June 2006 (crosses) and the corresponding Gigi volatility (continuous line), obtained through a *local* calibration as discussed in the text. We recall that $k \equiv \ln(1+\theta_i\mathcal{K})/(1+\theta_i L_i(t_0))$ with $L_i(t_0) = 3.986\%$ in the 30/360 day count convention; the strikes are in the range $2\% - 8\%$.

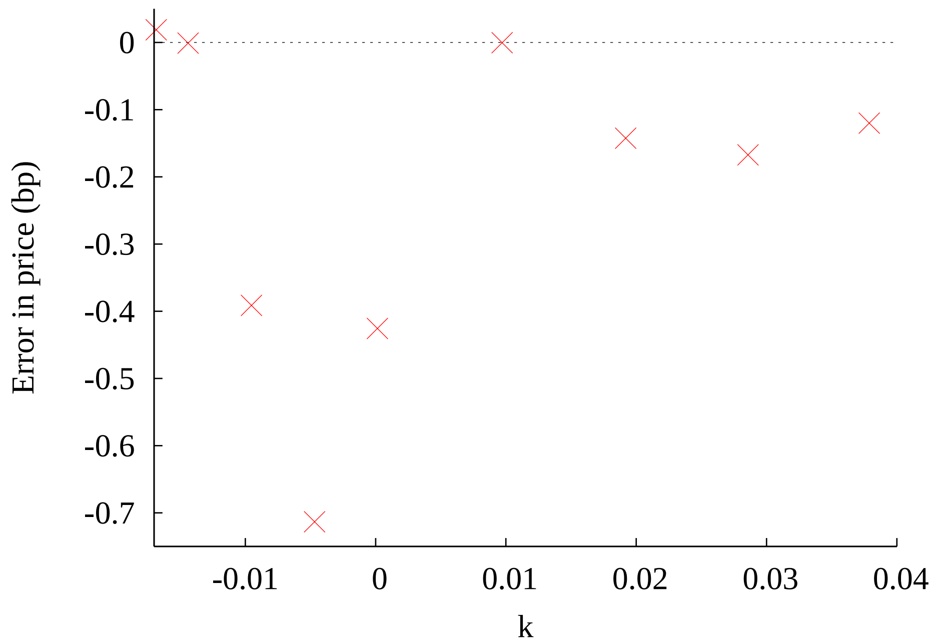

Figure 4: Differences (in bps) between market prices (value date on the 8^{th} June 2006) and Gigi prices calibrated as in figure 3.

In figure 4 we show the errors (in bps) computed as the differences between market prices and model prices calibrated as in figure 2. The *local* calibration approach allows to equate both the implied volatility and the slope around the relevant strike; in practice the error we get is negligible for $\mathcal{K} \geq 5\%$ and it is relatively small in the other cases for strikes in the range $2\% - 8\%$. This calibration is then very good for our aim; in any case if one wishes a better fit all over the strikes, even a *global* calibration approach is straightforward.

7 Conclusions

In this paper we have introduced Gigi model, a multifactor interest rate model that allows to manage in a simple way volatility smile. This model reminds the Uncertain Volatility Model approach [7] where the weights are provided through an elementary continuous distribution. One of the main advantages of Gigi model is that caplets and floorlets can be priced via one-dimensional closed formulas (9) and (10), through an approach that is reminiscent of the one in [23]. The other main advantage is that Monte Carlo simulations are elementary.

It is also useful to mention that $G_i g_i$ model can be of particular interest for its simplicity when underlyings belong to different asset classes as in the case of the so called Hybrids.

Gigi model satisfies three main properties:

- The dynamics between reset dates (equations (14) and (17)) is modeled as an elementary Markov chain whose transition probabilities are simple to be implemented numerically. This property holds even in the spot measure, so that exotic contingent claims can be priced easily through Monte Carlo simulations even in presence of callable features. We recall that this task is not straightforward in the standard Sabr model [13].

  Furthermore, the model is *Sticky Delta*, and then it reproduces the dynamics of the implied volatility curve where implied volatility and Libor rates move in the same direction [13].

- Calibration is straightforward due to the simplicity of the 1-dimensional formulas for caplets/floorlets (9) and (10) where the Laplace transform $\mathcal{L}_i$ is given by (16). The *local* calibration we propose in section **6** is particularly adequate when the exotic derivative is not only sensitive to the absolute level of the implied volatility but also to its local slope, i.e. the model properly reflects the digital risk related to the slope of the *implied volatility curve*.

- It is a parsimonious description of the implied volatility curve. We get a straight correspondence between model parameters and financial quantities, avoiding any redundancy issue. The model generalizes the caplet/floorlet Black formula with only one parameter (the implied volatility) to a scheme with 3 parameters associated to the characteristics of the implied volatility curve. Caplet/floorlet price has two parameters in addition to the average volatility: η_i that controls implied volatility skew, and κ_i that is related to the excess kurtosis of the log-price distribution.

A stochastic volatility model for interest rates should reproduce in a simple way cap/floor implied volatility and it should allow a simple numerical implementation: in this paper we have shown that Gigi model can be a good candidate to achieve this task.

Acknowledgments

I wish to thank Marc Chesney of the University of Zurich for his patience in introducing me to stochastic volatility, Peter Carr of Bloomberg for having formulated the question that has originated this paper, Claudio Albanese of the Imperial College for a nice discussion on Sabr models and *Sticky Delta* rule, Matthew Herring of Mitsubishi UFJ for his observations on *local* calibration, P.-H. Cheong of Deutsche Bank for his hint on a corrected-Black-like cumulated distribution when managing digital risk, Peter Laurence of the University of Rome for a constructive conversation on HJM framework and possible generalizations, Claudio Nordio and Lorenzo Giada of Banca Italease for several discussions on the numerical implementation of Gigi model. I thank my Father for a never-ending encouragement during this work. I also acknowledge the Finance Department of HEC (Paris) where this research has begun for its kind hospitality.

Appendix A

In this appendix we briefly recall some simple properties of change of measure in the discrete time case.

Let us suppose that $\xi(l)$ is a vector in $\Re^d$ of zero mean Gaussian random variables under the measure π s.t.

$$Corr\left(\xi_n(l), \xi_j(m)\right) = \rho_{n\,j}\;\delta_{l\,m} \quad .$$

A change of measure in this case is

$$\xi'(l) = \xi(l) + \sqrt{\hat{t}_{l+1} - \hat{t}_l}\;\rho\;\lambda(l)$$

with $\lambda(l)$ a constant vector in $\Re^d$ and the Radon-Nykodim derivative in the discrete case

$$\frac{d\pi'}{d\pi}(l) = exp\left\{-\left[\sqrt{\hat{t}_{l+1} - \hat{t}_l}\;\lambda(l)\cdot\xi(l) + \frac{\hat{t}_{l+1} - \hat{t}_l}{2}\lambda(l)\cdot\rho\;\lambda(l)\right]\right\}$$

is such that $\xi'(l)$ is a vector of zero mean Gaussian random variables under the measure π'.

The t_j-forward measure is such that

$$E_0^{(0)}\left[D_{0j}\;h\left(\{B_l(t_j)\}_{l=j,..,N}\right)\right] = B_{0j}(t_0)E_0^{(j)}\left[h\left(\{B_l(t_j)\}_{l=j,..,N}\right)\right] \tag{19}$$

where the D_{0j} is the discount factor between t_0 and t_j, and $B_{0j}(t_0)$ is the ZC bond in t_0 between the same dates; $E_0^{(j)}[\cdot]$ is the expectation under the j^{th} forward measure given the information at time t_0 and $E_0^{(0)}[\cdot]$ the expectation under the spot measure.

We introduce the function $m(t) : \Re \to \mathbb{N}$ such that

$$m(t) = m \;\; when \;\; t_{m-1} \leq t < t_m \;\; .$$

LEMMA:

In the BBMM, the change of measure between the spot and the t_j -forward measure is

$$\xi^{(j)}(l) = \xi^{(0)}(l) - \sqrt{\hat{t}_{l+1} - \hat{t}_l}\; \rho \sum_{n=m(\hat{t}_l)}^{j-1} \hat{\sigma}_n(l) \tag{20}$$

Proof:

In order to prove the lemma we have to show that, using the change of measure (20), the above property holds. Using model dynamics (4) and the change of measure (20), in the spot measure the discount D_{0j} is

$$\begin{aligned} D_{0j} = \;& \prod_{i=0}^{j-1} B_i(t_i) = \prod_{i=0}^{j-1} B_i(t_0)\; \exp\left\{ - \sum_{i=0}^{j-1} (t_i - t_0) \left[\frac{\eta_i}{2} v_i^2(t_0, t_i) + \psi_i \left[\frac{\eta_i}{2} \right] \right] \right\} \\ & \exp\left\{ -\frac{1}{2} \sum_{l:\, t_0 \leq \hat{t}_l < t_j} (\hat{t}_{l+1} - \hat{t}_l) \sum_{i=m(\hat{t}_l)}^{j-1} \hat{\sigma}_i(l) \cdot \rho \sum_{n=m(\hat{t}_l)}^{j-1} \hat{\sigma}_n(l) + \sum_{l:\, t_0 \leq \hat{t}_l < t_j} \sqrt{\hat{t}_{l+1} - \hat{t}_l} \sum_{i=m(\hat{t}_l)}^{j-1} \hat{\sigma}_i(l) \cdot \xi^{(0)}(l) \right\} \end{aligned}$$

Observing that the second line in the above expression is the Radon-Nycodim derivative between the spot and the t_j-forward measure, in the t_j-forward measure we can write

$$\begin{aligned} E_0^{(0)} \left[D_{0j}\; h\left(\{B_l(t_j)\}_{l=j,..,N} \right) \right] = \;& \frac{B_{0j}(t_0)}{\prod_{i=0}^{j-1} \mathcal{L}_i \left[\frac{\eta_i}{2}; s, t \right]} \\ & E_0^{(j)} \left[\exp\left\{ - \sum_{i=0}^{j-1} (t_i - t_0) \frac{\eta_i}{2} v_i^2(t_0, t_i) \right\} h\left(\{B_l(t_j)\}_{l=j,..,N} \right) \right] \end{aligned}$$

The lemma is proven after we notice that the random variables in the set $\{v_i^2\}_{i=0,..,j-1}$ are independent from the random variables in $h\left(\{B_l(t_j)\}_{l=j,..,N} \right)$.$\diamondsuit$

PROPOSITION:

In the t_j-forward measure ($j < i$) BBMM return is

$$
\begin{aligned}
x_i(s,t) = & -\sum_{l:\, s\le \hat{t}_l<t} \left(\hat{t}_{l+1}-\hat{t}_l\right)\hat{\sigma}_i(l)\cdot\rho \sum_{n=m(u)\wedge j}^{i-1}\hat{\sigma}_n(l) - \\
& \frac{1+\eta_i}{2}\sum_{l:\, s\le \hat{t}_l<t}\left(\hat{t}_{l+1}-\hat{t}_l\right)\hat{\sigma}_i(l)\cdot\rho\hat{\sigma}_i(l) + \sum_{l:\, s\le \hat{t}_l<t}\sqrt{\hat{t}_{l+1}-\hat{t}_l}\hat{\sigma}_i(l)\cdot\xi^{(i)}(l) - \mathcal{L}_i\left[\frac{\eta_i}{2};s,t\right]
\end{aligned}
\tag{21}
$$

or equivalently

$$
x_i(s,t) = -\int_s^t \sigma_i(u)\sum_{n=m(u)\wedge j}^{i-1}\cdot\rho\,\sigma_n(u)du - (t-s)\left[\frac{1+\eta_i}{2}v_i^2(s,t)+\psi_i\left[\frac{\eta_i}{2}\right]\right]+\sqrt{t-s}\,v_i(s,t)g_i(s,t)
$$

Proof:

Straightforward given the above lemma. $\diamondsuit$

Appendix B

In this appendix we prove the Lemma of section **4**.

Let us consider the $I_i(k)$ case in the t_i-forward measure. The integral can be written

$$
I_i(k) = \int_0^{+\infty} dv_i^2\wp\left[v_i^2\right]\int_{-\infty}^{+\infty}dx_i\wp\left[x_i|v_i^2\right]\left\{\begin{array}{cc} 1-H(x_i+k) & \eta_i>-1 \\ H(-k-x_i) & \eta_i<-1\end{array}\right.
$$

For $\eta_i/2\in(-1/2,+\infty)\cap\mathcal{A}$, given the conditional probability (6) and the definition of k'

$$
I_i(k) = 1-\int_0^{+\infty}dv_i^2\wp\left[v_i^2\right]\int_{-\infty}^{+\infty}dx_i\frac{1}{\sqrt{2\pi(t_i-t_0)v_i^2}}\exp\left\{-\frac{\left[x_i+\frac{1+\eta_i}{2}(t_i-t_0)v_i^2\right]^2}{2(t_i-t_0)v_i^2}\right\}H(x_i+k')
$$

Using the Gaussian relation

$$
\frac{1}{\sqrt{t}v_i}\exp\left(-\frac{x_i^2}{2tv_i^2}\right) = \int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\exp\left\{-\frac{tv_i^2\omega^2}{2}+ix\omega\right\} \qquad ,\forall t>0
$$

we can rewrite

$$
\begin{aligned}
I_i(k) \;\; = \;\; & 1-\int_0^{+\infty}dv_i^2\wp(v_i^2)\int_{-\infty}^{+\infty}dx_i e^{-x_i(1+\eta_i)/2}H(x_i+k') \\
& \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\exp\left\{-\frac{(t_i-t_0)v_i^2}{2}\left(\omega^2+(1+\eta_i)^2/4\right)+ix_i\omega\right\}
\end{aligned}
$$

Observing that for $\eta_i/2 \in (-1/2, +\infty) \cap \mathcal{A}$

$$\int_{-\infty}^{+\infty} dx_i H(x_i + k') \exp\{-x_i(1+\eta_i)/2 + ix_i\omega\} = -\frac{\exp\{-k'(i\omega - (1+\eta_i)/2)\}}{i\omega - (1+\eta_i)/2}$$

then, after using the Fubini theorem and some simple algebra, the proposition is proven for $\eta_i/2 \in (-1/2, +\infty) \cap \mathcal{A}$.

For $\eta_i/2 \in (-\infty, -1/2) \cap \mathcal{A}$ we have

$$I_i(k) = \int_0^{+\infty} dv_i^2 \wp\left[v_i^2\right] \int_{-\infty}^{+\infty} dx_i \frac{1}{\sqrt{2\pi(t_i - t_0)v_i^2}} \exp\left\{-\frac{\left[x_i + \frac{1+\eta_i}{2}(t_i - t_0)v_i^2\right]^2}{2(t_i - t_0)v_i^2}\right\} H(-k' - x_i)$$

Observing that

$$\int_{-\infty}^{+\infty} dx_i H(-k' - x_i) \exp\{-x_i(1+\eta_i)/2 + ix_i\omega\} = \frac{\exp\{-k'(i\omega - (1+\eta_i)/2)\}}{i\omega - (1+\eta_i)/2}$$

we have

$$I_i(k) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-k'[i\omega - (1+\eta_i)/2]}}{i\omega - (1+\eta_i)/2} \mathcal{L}_i\left[\frac{\omega^2 + (1+\eta_i)^2/4}{2}\right]$$

and then the proposition is proven for $I_i(k)$ and $\eta_i/2 \in \mathcal{A} \setminus \{-1/2\}$.

In order to prove the Lemma in the case $\eta_i = -1$, it can be useful to observe that for $\eta_i \neq -1$ the integral $I_i(k)$ is equal to

$$I_i(k) = \frac{1}{2} + \mathcal{P}\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-ik'\omega}}{i\omega} \mathcal{L}_i\left[\frac{\omega^2 - i(1+\eta_i)\omega}{2}\right]$$

where $\mathcal{P}$ is the Cauchy principal value of the integral. The value of $I_i(k)$ in $\eta_i = -1$ can be defined as the limit for $\lim_{\eta_i \to -1} I_i(k)$ or, as in standard analysis [24], we have to modify the integral domain in equation (8) slightly below the real axis.

Mutatis mutandis the proof is the same for the $J_i(k)$ case. $\diamondsuit$

Appendix C

In order to show that the results in sections **3** and **4** are not limited to the Gigi case, in this appendix we show another example in the BBMM class where the stochastic volatility is described by a Gamma variable.

In this case the vol between two dates in $\mathcal{T}$ is

$$v_i^2(\hat{t}_l, \hat{t}_{l+1}) = \mathcal{V}_i^2 G_i(l)$$

with $G_i(l)$ an unitary mean Gamma distributed random variable with variance $\kappa_i/(\hat{t}_{l+1} - \hat{t}_l)$. The vol between value date t_0 and fixing date t_i is then

$$v_i^2 = \mathcal{V}_i^2 G_i$$

with G_i an unitary mean Gamma distributed random variable with variance $\kappa_i/(t_i - t_0)$ and the Laplace transform for v_i is

$$\ln \mathcal{L}_i[\omega] = -\frac{t_i - t_0}{\kappa_i} \ln(1 + \mathcal{V}_i^2 \kappa_i \omega) \qquad .$$

with $\mathcal{A} = (-1/(\mathcal{V}_i^2 \kappa_i), +\infty)$.

All the results in sections **3** and **4** hold in this case whatever is the chosen partition $\mathcal{T}$: in particular caps and floors can be priced according to (9) and (10) with the above $\mathcal{L}_i$.

If we consider a single factor model for a generic time $t \leq t_i$ (and choosing $t_0 = 0$) the return (5) becomes

$$x_i(0,t) = -\frac{1+\eta}{2}\, t\, \mathcal{V}^2\, G^2 + \sqrt{t}\, \mathcal{V}\, G\, g + \frac{t}{\kappa} \ln\left(1 + \frac{\eta \mathcal{V}^2 \kappa}{2}\right)$$

with G an unitary mean Gamma distributed random variable with variance κ/t and g a standard normal random variable; therefore we get a (discrete) Variance Gamma model for a generic partition $\mathcal{T}$. In particular we get the same dynamics of [9] in the continuous time limit, i.e. when the maximal lag in the partition $\mathcal{T}$ $\max_l(\hat{t}_{l+1} - \hat{t}_l)$ goes to zero.

Appendix D

An IG random variable G with unitary mean and variance κ has

$$\wp\,[G]\, dG = \frac{1}{\sqrt{2\pi\, \kappa\, G^3}} \exp\left[-\frac{(G-1)^2}{2\, \kappa\, G}\right] dG \qquad .$$

In order to simulate the above IG variable, a simple algorithm that involves only one uniform random variable and one χ_1^2 variable is the following [21] :

- Generate independent (u, z) with $u \sim Uniform(0, 1)$ and $z \sim \chi_1^2$.
- Let $G^* \equiv 1 - \frac{\kappa}{2}\left(\sqrt{z^2 + \frac{4z}{\kappa}} - z\right)$
- If $(1 + G^*)u > 1$ then $G = 1/G^*$, else $G = G^*$.

Once we have shown that an IG variable is very simple and fast to generate, Gigi dynamics (14) between two reset dates t_l and t_{l+1} is straightforward to simulate:

1. Generate a vector of standard normal $\xi(l) \in \Re^d$ correlated through the correlation matrix ρ and a set of N independent IG variables $G_i(l)_{i=1,..,N}$.
2. The Gaussian variables in the dynamics are $g_i(l) = \dfrac{\nu_i \xi(l)}{\mathcal{V}_i}$ and the volatilities are $v_i(l) = \mathcal{V}_i G_i(l)$.

The above algorithm shows the simplicity of Monte Carlo dynamics in the Gigi model. This clarifies the advantage of considering the additional set of the hidden variables v_i instead of modeling directly the (not-Gaussian) x_i: in an elementary way we get log-prices with correlated dynamics and with fat tails distributions.

References

[1] O.E. Barndorff-Nielsen *Processes of Normal Inverse Gaussian type*, Finance Stoch. **2** (1998), 41-68.

[2] R. Baviera, *Bond Market Model*, International Journal of Theoretical and Applied Finance **9** (2006), 577-596.

[3] R. Baviera and L. Giada, *A perturbative approach to Bermudan Options pricing*, Available at: ssrn.com/abstract=941318 (2006).

[4] R. Baviera, *A simple solution for Sticky Cap and Sticky Floor*, Quantitative Finance **7** (2007), 285-287.

[5] F. Black, *The pricing of commodity contracts*, Journal of Financial Economics **3** (1976), 167-179.

[6] D. Brigo and F. Mercurio, *Interest Rate Models: Theory and Practice*, Springer Finance, Heidelberg (2006).

[7] D. Brigo, F. Mercurio and F. Rapisarda, *Smile at the Uncertainty*, Risk **5** (2004), 97-101.

[8] P. Carr and D.B. Madan, *Option valuation using the fast Fourier transform*, Journal of Computational Finance **2** (1998), 61-73.

[9] P. Carr, D.B. Madan and E.C. Chang, *The Variance Gamma Process and Option Pricing*, European Finance Review **2** (1998), 79-105.

[10] E. Derman, *Regimes of volatility*, Risk **4** (1999), 55-59.

[11] E. Eberlein and F. Őzkan, *The Lévy Libor model*, Finance Stoch. **9** (2005), 327-348.

[12] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer Finance, New York (2003).

[13] P.S. Hagan, D. Kumar, A.S. Lesniewski, and D. Woodward *Managing Smile Risk*, Wilmott magazine (2002), 84-108.

[14] M. Henrard, *Skewed Libor market model and Gaussian HJM explicit approaches to rolled deposit options*, Journal of Risk **9** (2007), 95-116.

[15] S.L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies, **6** (1993), 327-343.

[16] J. C. Hull and A. White, *The Pricing of Options on Assets with Stochastic Volatilities*, Journal of Finance **42** (1987), 281-300.

[17] R.W. Lee, *The Moment Formula for Implied Volatility at Extreme Strikes*, Mathematical Finance **14** (2004), 469-480.

[18] D.B. Madan and E. Seneta, *The variance gamma (V.G.) model for share market returns*, Journal of Business **63** (1990), 511-524.

[19] D.B. Madan and F. Milne, *Option pricing with VG martingale components*, Mathematical Finance **1** (1991), 39-55.

[20] F. Mercurio, *A Multi-Stage Uncertain Volatility Model*, Available at: www.fabiomercurio.it (2002).

[21] J.R. Michael, W.R. Schucany and R.W. Haas, *Generating random variates using transformations with multiple roots*, American Statistician **30-2** (1976), 88-90.

[22] R. Rebonato, *Modern Pricing of Interest-Rate Derivatives*, Princeton University Press, Princeton (2002).

[23] A. Skabelin, *Pricing of options on assets with level dependent stochastic volatility*, in *Noise and Fluctuations in Econophysics and Finance*, Proc. SPIE **5848**, Eds. D. Abbott, J.P. Bouchaud, X. Gabaix and J.L. McCauley (2005), 110-124.

[24] A. G. Svesnikov and A. N. Tichonov, *Teorija funkcij kompleksnoj peremennoj*, Nauka, Moscow (1974).