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BOND MARKET MODEL*

ROBERTO BAVIERA
Abaxbank, corso Monforte, 34
I-20122 Milan, Italy
roberto.baviera@abaxbank.com

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We describe the Bond Market Model, a multi-factor interest rate term structure model, where it is possible to price with Black-like formulas the three classes of over-the-counter plain vanilla options. We derive the prices of caps/floors, bond options and swaptions. A comparison with Libor Market Model and Swap Market Model is discussed in detail, underlining advantages and limits of the different approaches.

Keywords: HJM framework; term structure model; caps/floors; bond options; swaptions.

1. Introduction

The interest rate option market is the world's largest options market. The Bank of International Settlements reports, for the first semester 2004 [1] relative to the over-the-counter (OTC) market of interest rate options, an outstanding of approximatively 24 trillions in US-dollar terms; by far the largest part of this volume is due to the three classes of liquid plain vanilla OTC interest rate options: caps/floors, bond options and swaptions.

Each of these derivatives are traded as separate products and valued with closed formulas derived in the Black framework [4]. The market convention is to quote each derivative in terms of the implied volatility which sets the Black model price equal to the market price. Note that the convention of quoting prices in terms of the implied volatility from the Black model does not necessarily mean that market participants view the Black model as the most appropriate model. Rather, implied volatilities are simply a more convenient way of quoting prices when only one class of derivative is traded, since implied volatility tends to be more stable over time than the actual price at which the interest rate option would be traded. Yet it is well known (see, e.g., [6, 17, 19, 20]) that the assumption leading to the three market-model-formulas are not compatible.

*The views expressed in this paper are those of the author and do not necessarily reflect those of the bank.

When exotic derivatives are involved, however, the option can insist on different parts of the term structure. From a practitioner's perspective it is crucial to have a model for the term structure as a whole with a fast and accurate calibration on the liquid plain vanilla OTC derivatives; it can be also important to have some analytical formulas for a set of exotic options.

From a theoretical perspective, a unified model for the interest rate curve with a good fit of market prices (whatever is the number of state variables involved) can provide new insights on the entire term structure dynamics.

The Bond Market Model (BMM in the following) is a multi-factor Gaussian Heath–Jarrow–Morton model [7, 10] in the form introduced in [2]. It has been already shown that for some exotics the BMM can provide analytical closed form solutions [2, 3]. The aim of this paper is to show, how in a multi-factor stochastic interest rate model, is possible to price with Black-like formulas the three classes of OTC fixed income options. The BMM provides a unified framework for valuing fixed-income derivatives with the important advantage to provide a possible calibration in an easy way.

The multi-factor models used by practitioners are particular realization of the Heath–Jarrow–Morton model; the two mostly used, the Libor Market Model [5, 14, 16] (LMM in the following) and the Swap Market Model [13] (SMM), present different advantages in plain vanilla pricing. In LMM, the Libor rate is lognormal (under a particular measure [5]) and cap and floor prices have the same form of Black formula. However, it becomes difficult to price swaptions and then to calibrate correlations between different Libor rates (see [6] for a detailed discussion); no explicit solution for bond options is known. The Swap Market Model leads to a Black formulation of the swaption price but no pricing formula is known for caps and floors.

We also describe the specific feature in interest rate derivatives that relates the BMM, Black models used by market participants, LMM and SMM. We stress advantages and limits of each model.

An outline of the paper is as follows. In the next section we describe the Heath–Jarrow–Morton framework and we introduce the Bond Market Model in Sec. 3. The pricing formulas of the three different classes of OTC plain vanilla interest rate derivatives are deduced in Sec. 4. We compare BMM with the Libor Market Model and the Swap Market Model in Sec. 5, we discuss in detail each approach and the relation with Black formulas. Finally in Sec. 6 we state some concluding remarks.

2. Heath–Jarrow–Morton Framework

In this section we introduce the notation and the Heath–Jarrow–Morton framework.

Define today as the value date T_0, we introduce a collection of reset dates $\{T_i\}_{i=0,\ldots,N}$, starting from the value date and such that $T_0 < T_1 < \cdots < T_N$ and with the lag $\theta_i \equiv T_{i+1} - T_i$ not greater than one year. There is an equivalency to consider

- $L_i(t)$, the forward Libor rate at time t [17] of the Libor Rate $L_i(T_i)$ between T_i and T_{i+1}, fixing in T_i and payment in T_{i+1},
- $B_i(t)$, the forward price in t of the zero coupon bond (ZC in the following) starting in T_i which pays 1 in T_{i+1}, i.e. in the standard notation

$$B_i(t) = \frac{B(t, T_{i+1})}{B(t, T_i)},$$

  with $B(t, T)$ the value in t of the ZC which pays 1 in T.

The two sets of quantities are equivalent since the following relation holds (see, e.g., [6, 17, 19]):

$$L_i(t) = \frac{1}{\theta_i}\left(\frac{1}{B_i(t)} - 1\right). \tag{2.1}$$

The Heath–Jarrow–Morton approach to interest rate modeling [10] allows to describe in an arbitrage free setting the dynamics of the term structure.

Let $(\Omega, \mathcal{F}, \mathcal{Q})$, with $\{\mathcal{F}_t : T_0 \leq t \leq T_N\}$, be a complete filtered probability space satisfying the usual hypothesis, the Heath–Jarrow–Morton model assumes that, under the risk-neutral measure, the dynamics for the instantaneous forward rate $f(t, T)$ between t and $T \leq T_N$ is

$$df(t, T) = \frac{1}{2}\frac{\partial}{\partial T}\sigma(t, T)\rho\sigma(t, T)dt - \frac{\partial}{\partial T}\sigma(t, T)dW(t), \tag{2.2}$$

where $\sigma(t, T)$ is an M-dimensional vector of adapted processes (in particular in the Gaussian case $\sigma(t, T)$ is a deterministic function of time) with $\sigma(T, T) = 0$ and W is an M-dimensional Brownian motion with instantaneous covariance $\rho = (\rho_{ij=1,\dots,M})$

$$dW_i(t)dW_j(t) = \rho_{ij}dt,$$

and $\sigma(t, T)dW(t)$ is the scalar product in $\Re^M$ between the two vectors $\sigma(t, T)$ and $dW(t)$. The integer M is the number of factors of the Heath–Jarrow–Morton model considered.

Equation (2.2) is equivalent to (see, e.g., [7])

$$dB(t, T) = B(t, T)[r_t dt + \sigma(t, T)dW(t)], \tag{2.3}$$

where the instantaneous rate

$$r_t = f(T_0, t) + \frac{1}{2}\int_{T_0}^{t}\frac{\partial}{\partial t}\sigma(t', t)\rho\sigma(t', t)dt' - \int_{T_0}^{t}\frac{\partial}{\partial t}\sigma(t', t)dW(t').$$

3. The Model

In this section we first deduce some properties shared by a large class of Heath–Jarrow–Morton models, then introduce the BMM as a particular subclass and derive the dynamics for ZCs according to this model.

Let us define the volatility

$$v_i(t) \equiv \sigma(t, T_{i+1}) - \sigma(t, T_i),$$

and impose the condition that

$$v_i(t) = 0 \quad \text{for } t \geq T_i. \tag{3.1}$$

We also introduce the function $k(t) : \Re \to \mathbb{N}$ such that

$$k(t) = k \quad \text{when } T_k \leq t < T_{k+1}.$$

With empty sums denoting zero the following lemma holds.

Lemma 3.1. *The dynamics for $B_i(t)$ is*

$$dB_i(t) = B_i(t) v_i(t) \left[- \sum_{j=k(t)+1}^{i-1} \rho v_j(t) dt + dW(t) \right]. \tag{3.2}$$

Proof. Starting from Eq. (2.3), using the definitions of $B_i(t)$ and $v_i(t)$ and condition (3.1), it is a straightforward application of Ito's Lemma. □

Lemma 3.1 describes the dynamics of the forward ZC B_i in the Heath–Jarrow–Morton model under the additional condition (3.1). In this dynamics v_i is the volatility of B_i, where B_i is the forward ZC bond with fixing in T_i; choosing $v_i(t)$ equal to zero after T_i is equivalent to impose that B_i ends its dynamics in T_i. The form of the drift term in (3.2) comes from the condition of no arbitrage, and it is a direct consequence of the Heath–Jarrow–Morton framework.

A driftless dynamics is obtained in the T_i-forward measure introduced in [9]; we show this result in the following Lemma 3.2, that is just a different rewriting of the previous one.

Lemma 3.2. *In the T_i-forward measure the dynamics becomes*

$$dB_i(t) = B_i(t) v_i(t) dW^{\{(i)\}}(t), \tag{3.3}$$

where $dW^{\{(i)\}}(t)$ is a Brownian motion under the forward measure Q^i.

Proof. It is enough to rewrite (3.2) in the T_i-forward measure [9]. □

The dynamics is similar when considering forward start ZC bond with longer tenor. With empty sums denoting zero and empty products denoting 1, let us define the forward ZC bond which starts in T_α and pays 1 in T_ω

$$B_{\alpha\omega}(t) \equiv \prod_{i=\alpha}^{\omega-1} B_i(t),$$

and the volatility

$$v_{\alpha\omega}(t) \equiv \begin{cases} \displaystyle\sum_{i=\alpha}^{\omega-1} v_i(t) & \text{if } t \le T_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.3. *In the T_α-forward measure the dynamics for $B_{\alpha\omega}(t)$ is*

$$dB_{\alpha\omega}(t) = B_{\alpha\omega}(t) v_{\alpha\omega}(t) dW^{\{(\alpha)\}}(t). \tag{3.4}$$

Proof. Just an application of Ito's lemma using the above definitions and Eq. (3.3). □

In this paper we show that the role of volatility $v_i(t)$ is crucial in interest rate derivatives. Let us notice that, in Eqs. (3.3) and (3.4), the forward start ZC bonds are Markov processes if the volatility is a Markov process itself (and in particular when it is a deterministic function of time). Volatility $v_i(t)$ is a Markov process in the interest rate models mostly used in practice, e.g., LMM and SMM are in this subclass of Heath–Jarrow–Morton models. This fact implies that all financial assets we consider in this paper are Markov processes, even if the instantaneous rate r_t is not (in general) a Markov process.[1]

In the Bond Market Model we chose $v_i(t)$ as a deterministic function of time, therefore it is a version of the multi-factor Gaussian Heath–Jarrow–Morton model [7, 17] and then the forward start ZC bonds are also lognormal. In the following (unless explicitly stated otherwise) we restrict our attention to a deterministic volatility $v_i(t)$.

4. OTC Plain Vanilla Interest Rate Derivatives

This section shows how to price the three different OTC plain vanilla with closed formulas in the BMM. We first show pricing formulas for ZC bonds: this is a generalization of the well known result obtained in [12]. We deduce the price of cap/floor and the exact formula for coupon bearing bond option, we then show how to obtain a Black formula that is an excellent approximation. Finally the last part of this section is devoted to describe swaption solution using a relation with coupon bond option.

4.1. *European zero coupon option*

At the expiry T_α an European call (put) option gives the right to buy (sell) the ZC $B_{\alpha\omega}$ at the strike price K.

The call option is equal to

$$\mathcal{C}^{(B)}_{\alpha\omega}(T_0) \equiv E[e^{-\int_{T_0}^{T_\alpha} r_t dt}(B_{\alpha\omega}(T_\alpha) - K)^+],$$

[1] We thank one referee for having underlined this property.

where the superscript $^+$ denotes the maximum between zero and the argument, and $E[\cdot]$ the $\mathcal{F}_0$ conditional expectation operator, the put option is

$$\mathcal{P}^{(B)}_{\alpha\omega}(T_0) \equiv E[e^{-\int_{T_0}^{T_\alpha} r_t dt}(K - B_{\alpha\omega}(T_\alpha))^+].$$

The following result is a straightforward generalization of [12].

Proposition 4.1. *In the BMM the call option is*

$$\mathcal{C}^{(B)}_{\alpha\omega}(T_0) = B_{0\alpha}(T_\alpha)\big[B_{\alpha\omega}(T_0)N\big(d_1^{(B)}\big) - KN\big(d_2^{(B)}\big)\big], \tag{4.1}$$

and the put is

$$\mathcal{P}^{(B)}_{\alpha\omega}(T_0) = B_{0\alpha}(T_\alpha)\big[KN\big(-d_2^{(B)}\big) - B_{\alpha\omega}(T_0)N\big(-d_1^{(B)}\big)\big], \tag{4.2}$$

where $N(\cdot)$ is the cumulative Gaussian distribution and

$$d_1^{(B)} = \frac{1}{\nu_{\alpha\omega}\sqrt{T_\alpha - T_0}} \ln\frac{B_{\alpha\omega}(T_0)}{K} + \frac{1}{2}\nu_{\alpha\omega}\sqrt{T_\alpha - T_0}\,,$$
$$d_2^{(B)} = \frac{1}{\nu_{\alpha\omega}\sqrt{T_\alpha - T_0}} \ln\frac{B_{\alpha\omega}(T_0)}{K} - \frac{1}{2}\nu_{\alpha\omega}\sqrt{T_\alpha - T_0}$$

and

$$\nu^2_{\alpha\omega} \equiv \frac{1}{T_\alpha - T_0}\int_{T_0}^{T_\alpha} v_{\alpha\omega}(t)\rho v_{\alpha\omega}(t)dt.$$

Proof. Let us consider the ZC call case, *mutatis mutandis* the put case is equivalent. Writing the ZC call option in the T_α-forward measure, one obtains

$$\mathcal{C}^{(B)}_{\alpha\omega}(T_0) \equiv B_{0\alpha}(T_0)E^{(\alpha)}[B_{\alpha\omega}(T_\alpha) - K]^+.$$

Since $B_{\alpha\omega}$ has a dynamics described in Eq. (3.4), the proposition follows, as in [15], for an underlying dynamics martingale and lognormal with a time dependent volatility. □

Equations (4.1) and (4.2) show that in the BMM an option on a ZC bond $B_{\alpha\omega}(T_\alpha)$ is priced according to the Black formula for ZC bond options. In the following we show that a similar result holds also for the coupon bearing bond case. We have named the Bond Market Model in this way since the bond option reduces to the Black formula.

4.2. *Cap and floor*

In a plain vanilla cap (floor) there are N possible payments each one related to an option called caplet (floorlet). The payoff of the ith caplet is established at time T_i as the difference, if positive, between the Libor rate with fixing at time T_i and a strike K. The ith payoff is established in T_i, calculated for the lag θ_i and paid in T_{i+1}.

A cap is equal to

$$C(T_0) \equiv \sum_{i=1}^{N} c_i(T_0),$$

and a floor is

$$F(T_0) \equiv \sum_{i=1}^{N} f_i(T_0).$$

The ith plain vanilla caplet with maturity in T_i is

$$c_i(T_0) \equiv \theta_i E[e^{-\int_{T_0}^{T_{i+1}} r_t dt}(L_i(T_i) - K)^+], \tag{4.3}$$

and the ith floorlet with strike K is

$$f_i(T_0) \equiv \theta_i E[e^{-\int_{T_0}^{T_{i+1}} r_t dt}(K - L_i(T_i))^+]. \tag{4.4}$$

Proposition 4.2. *In the BMM model caplet and floorlet prices are*

$$\begin{aligned} c_i(T_0) &= B_{0i+1}(T_0)\big[(1+\theta_i L_i(T_0))N\big(d_1^{(L)}\big) - (1+\theta_i K)N\big(d_2^{(L)}\big)\big], \\ f_i(T_0) &= B_{0i+1}(T_0)\big[(1+\theta_i K)N\big(-d_2^{(L)}\big) - (1+\theta_i L_i(T_0))N\big(-d_1^{(L)}\big)\big], \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} d_1^{(L)} &= \frac{1}{\nu_i\sqrt{T_i - T_0}} \ln\frac{1+\theta_i L_i(T_0)}{1+\theta_i K} + \frac{1}{2}\nu_i\sqrt{T_i - T_0}\,, \\ d_2^{(L)} &= \frac{1}{\nu_i\sqrt{T_i - T_0}} \ln\frac{1+\theta_i L_i(T_0)}{1+\theta_i K} - \frac{1}{2}\nu_i\sqrt{T_i - T_0}\,, \end{aligned}$$

and

$$\nu_i^2 \equiv \frac{1}{T_i - T_0}\int_{T_0}^{T_i} v_i(t)\rho v_i(t)dt.$$

Proof. Let us prove the proposition for a caplet, *mutatis mutandis* the proof is the same for a floorlet. Using the property of conditional expectations

$$E[\cdot] = E[E[\,\cdot\,|\mathcal{F}_{T_i}]],$$

and that (see, e.g., [7])

$$B_i(T_i) = E[e^{-\int_{T_i}^{T_{i+1}} r_t dt}\,|\,\mathcal{F}_{T_i}],$$

caplet definition can be rewritten

$$c_i(T_0) = E[e^{-\int_{T_0}^{T_i} r_t dt} B_i(T_i)[(1+\theta_i L_i(T_i)) - (1+\theta_i K)]^+].$$

Using relation (2.1), in the T_i-forward measure, the above equation becomes

$$c_i(T_0) = (1+\theta_i K)B_{0i}(T_0)E^{(i)}\left[\frac{1}{1+\theta_i K} - B_i(T_i)\right]^+,$$

and then the proposition is proven as is the ZC option case. ◻

4.3. *Coupon bond option*

The coupon bond with coupon c, which starts in T_α, has $\omega - \alpha$ periods and pays 1 in T_ω is

$$P_{\alpha\omega}(c;t) \equiv c \sum_{i=\alpha+1}^{\omega} \theta_{i-1} B_{\alpha i}(t) + B_{\alpha\omega}(t).$$

In the T_α-forward measure the dynamics for $P_{\alpha\omega}(c;t)$ is

$$dP_{\alpha\omega}(c;t) = P_{\alpha\omega}(c;t) V_{\alpha\omega}(c;t) dW^{\{(\alpha)\}}(t), \tag{4.6}$$

where

$$V_{\alpha\omega}(c;t) \equiv \sum_{i=\alpha+1}^{\omega} \gamma_i^{\alpha\omega}(t) v_{\alpha i}(t),$$

and the weights $\gamma_i^{\alpha\omega}$ are

$$\gamma_i^{\alpha\omega}(t) \equiv \frac{1}{c\sum_{r=\alpha+1}^{\omega} \theta_{r-1} B_{\alpha r}(t) + B_{\alpha\omega}(t)} \cdot \begin{cases} c\theta_{i-1} B_{\alpha i}(t) & \forall i \neq \omega \\ (1 + c\theta_{\omega-1}) B_{\alpha\omega}(t) & i = \omega \end{cases}.$$

We observe that

$$\sum_{i=\alpha+1}^{\omega} \gamma_i^{\alpha\omega}(t) = 1,$$

and

$$\gamma_\omega^{\alpha\omega}(t) \gg \gamma_i^{\alpha\omega}(t) \quad \forall i \neq \omega.$$

A European call (put) option, at the expiry T_α, gives the right to buy (sell) the coupon bond $P_{\alpha\omega}(c)$ at the strike price K.

The call option is equal to

$$\begin{aligned}\mathcal{C}_{\alpha\omega}^{(e)}(T_0) &\equiv E[e^{-\int_{T_0}^{T_\alpha} r_t dt}(P_{\alpha\omega}(c;T_\alpha) - K)^+] \\ &= B_{0\alpha}(T_0) E^{(\alpha)}[P_{\alpha\omega}(c;T_\alpha) - K]^+,\end{aligned}$$

and the put option is

$$\begin{aligned}\mathcal{P}_{\alpha\omega}^{(e)}(T_0) &\equiv E[e^{-\int_{T_0}^{T_\alpha} r_t dt}(K - P_{\alpha\omega}(c;T_\alpha))^+] \\ &= B_{0\alpha}(T_0) E^{(\alpha)}[K - P_{\alpha\omega}(c;T_\alpha)]^+.\end{aligned}$$

Proposition 4.3. *In the BMM the call and put options are*:

$$\begin{aligned}\mathcal{C}_{\alpha\omega}^{(e)}(T_0) = B_{0\alpha}(T_0) E^{\mathcal{Q}}\Bigg[& c \sum_{i=\alpha+1}^{\omega} \theta_{i-1} B_{\alpha i}(T_0) \\ & \times \mathrm{Exp}\left(-\frac{1}{2}\nu_{\alpha i}^2 (T_\alpha - T_0) + \nu_{\alpha i}\sqrt{T_\alpha - T_0}\xi_i\right) \\ & + B_{\alpha\omega}(T_0)\mathrm{Exp}\left(-\frac{1}{2}\nu_{\alpha\omega}^2 (T_\alpha - T_0) + \nu_{\alpha\omega}\sqrt{T_\alpha - T_0}\xi_\omega\right) - K\Bigg]^+,\end{aligned} \tag{4.7}$$

$$\mathcal{P}^{(e)}_{\alpha\omega}(T_0) = B_{0\alpha}(T_0)E^{\mathcal{Q}}\Bigg[K - c\sum_{i=\alpha+1}^{\omega}\theta_{i-1}B_{\alpha i}(T_0)$$
$$\times \operatorname{Exp}\left(-\frac{1}{2}\nu^2_{\alpha i}(T_\alpha - T_0) + \nu_{\alpha i}\sqrt{T_\alpha - T_0}\xi_i\right)$$
$$- B_{\alpha\omega}(T_0)\operatorname{Exp}\left(-\frac{1}{2}\nu^2_{\alpha\omega}(T_\alpha - T_0) + \nu_{\alpha\omega}\sqrt{T_\alpha - T_0}\xi_\omega\right)\Bigg]^+, \tag{4.8}$$

with, under the Gaussian measure $\mathcal{Q}$, ξ_i is a zero mean unitary variance Gaussian variable such that

$$\operatorname{Corr}(\xi_i, \xi_j) \equiv \frac{1}{\nu_{\alpha i}\nu_{\alpha j}}\frac{1}{T_\alpha - T_0}\int_{T_0}^{T_\alpha} v_{\alpha i}(t)\rho v_{\alpha j}(t)dt.$$

Proof. It is enough to show that

$$B_{\alpha i}(T_\alpha) = B_{\alpha i}(T_0)\operatorname{Exp}\left(-\frac{1}{2}\nu^2_{\alpha i}(T_\alpha - T_0) + \nu_{\alpha i}\sqrt{T_\alpha - T_0}\xi_i\right),$$

is straightforward, given the dynamics (3.4). □

In the practice, the above result is never used since a very good approximation of the volatility is given by

$$\tilde{V}_{\alpha\omega}(c;t) \equiv \sum_{i=\alpha+1}^{\omega}\gamma_i^{\alpha\omega}(T_0)v_{\alpha i}(t).$$

In this case the following result holds.

Theorem 4.1. *In the BMM model, when the coupon bond volatility is approximated by $\tilde{V}_{\alpha\omega}$, the price of a call option with maturity T_α on the coupon bearing bond $P_{\alpha\omega}(c;T_\alpha)$ and strike K is*

$$\mathcal{C}^{(a)}_{\alpha\omega}(T_0) = B_{0\alpha}(T_0)\left[P_{\alpha\omega}(c;T_0)N\left(d_1^{(P)}\right) - KN\left(d_2^{(P)}\right)\right], \tag{4.9}$$

and the put option is

$$\mathcal{P}^{(a)}_{\alpha\omega}(T_0) = B_{0\alpha}(T_0)\left[KN\left(-d_2^{(P)}\right) - P_{\alpha\omega}(c;T_0)N\left(-d_1^{(P)}\right)\right], \tag{4.10}$$

where

$$d_1^{(P)} = \frac{1}{\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0}}\ln\frac{P_{\alpha\omega}(c;T_0)}{K} + \frac{1}{2}\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0},$$
$$d_2^{(P)} = \frac{1}{\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0}}\ln\frac{P_{\alpha\omega}(c;T_0)}{K} - \frac{1}{2}\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0},$$

and

$$\mathcal{V}^2_{\alpha\omega} \equiv \frac{1}{T_\alpha - T_0}\int_{T_0}^{T_\alpha}\tilde{V}_{\alpha\omega}(c;t)\rho\tilde{V}_{\alpha\omega}(c;t)dt.$$

Proof. Observing that the coupon bond dynamics is described by Eq. (4.6) and using the approximation $\tilde{V}_{\alpha\omega}(c;t)$ of the bond volatility at time t, the result follows as in the ZC case. □

Let us stop and comment. Equations (4.9) and (4.10) are the main result of the paper. We show that a coupon bond option can be viewed as a basket option of ZCs with different durations and priced in BMM according to a Black formula.

As in equity markets, where the Black formula is used by practitioners to price both single stock options and basket options traded in the market (e.g., options on SP500 or EuroStoxx50), we show that a similar approach can be followed in fixed income markets. The main difference is that, while in the former the weights of the assets are chosen equal, in the latter the weight of the ZC with the longest duration is much larger (even of two orders of magnitude) with respect to the others.

This fact allows (4.9) and (4.10) to be an excellent approximation of (4.7) and (4.8). In the remaining part of this subsection we show the quality of the proposed approximation.

To check this approximation let us consider an N factor model where only the ith component of the volatility $v_i(t)$ is different from zero; with this choice, each lag θ_i of the interest rate term structure has an evolution modeled by 1 factor of the curve dynamics.

Furthermore we consider for simplicity the case where volatility is time independent. Then $(v_i(t))_l = \nu_i \delta_{il}$ where δ_{il} is Dirichlet delta.

We consider, e.g., the Euro market on Friday, January 14, 2005, at 11:15 a.m. Frankfurt time. In Table 1 we report the forward zero coupons $B_i(T_0)$ and the values of volatilities ν_i.

The correlation matrix is chosen as

$$\rho_{ij} = e^{-a|T_i - T_j|},$$

with $a = 8 \cdot 10^{-2}$.

First, we need to check the quality of the approximation for a set of liquid ATM bond option straddles (call + put) prices with different maturities and tenors; ATM bond option straddles are equivalent to the corresponding ATM swaption straddles, as we show at the end of next subsection. In Table 2 we report for each straddle the price (in %) and the error, which is defined as the difference in bps[2] between the exact solution [see Eqs. (4.7) and (4.8)] and the Black formula [see Eqs. (4.9) and (4.10)]. One can conclude that for most bond option straddles the error is of the order of the hundredth of basis point, while for the 10y into 10y straddle, the liquid option with longest expiry and underlying tenor in the market, the error is of the order of the tenth of basis point. These errors are much lower than the typical bid-ask spreads and negligible in practice.

[2]One basis point (bp) is equal to 10^{-4}.

Table 1. Forward zero coupons $B_i(T_0)$ and the values of volatilities ν_i (in %) in the Euro market on Friday, January 14, 2005, at 11:15 a.m. Frankfurt time.

T_i (in Years)	$B_i(T_0)$	ν_i (in %)
0	0.97771	—
1	0.97284	0.636
2	0.96969	0.695
3	0.96661	0.709
4	0.96425	0.711
5	0.96171	0.699
6	0.95959	0.707
7	0.95781	0.684
8	0.95647	0.673
9	0.95605	0.665
10	0.95564	0.635
11	0.95510	0.623
12	0.95458	0.600
13	0.95406	0.574
14	0.95337	0.551
15	0.95346	0.554
16	0.95298	0.549
17	0.95287	0.540
18	0.95292	0.530
19	0.95286	0.524

Table 2. Prices and valuation errors for At The Money (ATM) liquid straddle (call + put) bond options (equivalent, as specified in the text, to ATM straddle swaptions). Prices (in percentage of the outstanding) are the sum of the exact solution for a call (4.7) and for a put (4.8), both computed with 10^7 random paths. Errors are the differences (in bps of the outstanding) for the straddle between the exact solutions (4.7) and (4.8) and the Black approximated formulas (4.9) and (4.10).

Swaption	Price (in %)	Error (in bp)
1y2y	1.003	−0.009
1y5y	2.367	−0.034
1y10y	3.986	−0.069
2y2y	1.454	−0.012
2y5y	3.308	−0.051
2y10y	5.446	−0.111
5y2y	2.073	−0.017
5y5y	4.555	−0.085
5y10y	7.260	−0.222
10y2y	2.116	−0.018
10y5y	4.506	−0.094
10y10y	7.246	−0.317

Second, in order to verify the behavior of the approximation with different strikes, we price a 5y-5y bond option with coupon $c = 4.341\%$, equal to the 5y-5y forward swap. We show in Fig. 1 the values obtained by the exact solution (4.7) of the call bond option as a function of the strike K. In Fig. 2 we plot

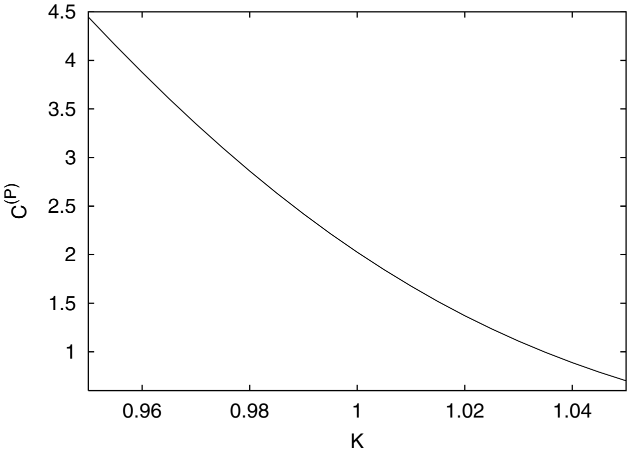

Fig. 1. We show the price of a 5y-5y call option on a bond with a coupon of $c = 4.341\%$ as a function of the strike K. The values (in percentage of the outstanding) are obtained via the exact solution (4.7) with 10^7 random paths.

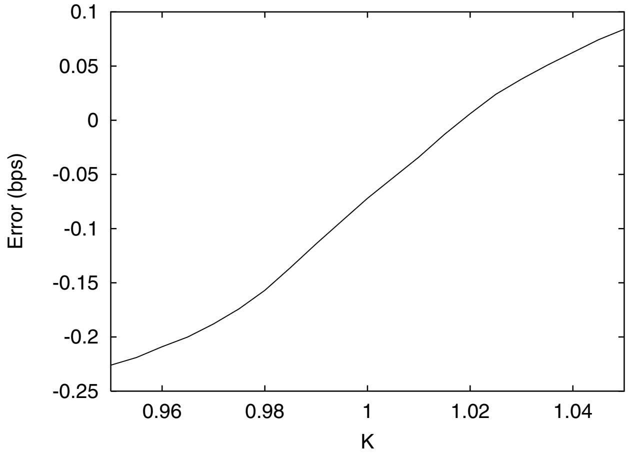

Fig. 2. We plot the differences in bps between the exact solution (4.7) and the Black approximated formula (4.9). The differences appear to be of the order of one tenth of bp. The two solutions are then equivalent for all practical purposes since the difference is negligible compared with the bid-ask spread present in the OTC market.

the differences in bps between the exact solution (4.7) and the Black formula (4.9). Compared with the bid-ask spread present in the OTC market (5 to 10 bps for at-the-money options and even the double for out-the-money options) the differences appear to be negligible. The two solutions are then equivalent for all practical purposes.

4.4. *Swaption*

A payer swaption with strike K_S and maturity T_α gives to the owner the right to enter in an interest rate swap where he pays the fixed leg with a rate K_S and receives the floating leg

$$\begin{aligned}\mathbf{PS}_{\alpha\omega}(T_0) &\equiv E[e^{-\int_{T_0}^{T_\alpha} r_t dt} BPV_{\alpha\omega}(T_\alpha)(S_{\alpha\omega}(T_\alpha) - K_S)^+] \\ &= B_{0\alpha}(T_0)E^{(\alpha)}[BPV_{\alpha\omega}(T_\alpha)(S_{\alpha\omega}(T_\alpha) - K_S)^+],\end{aligned}$$

where at time t the T_α-forward swap rate is

$$S_{\alpha\omega}(t) \equiv \frac{1 - B_{\alpha\omega}(t)}{BPV_{\alpha\omega}(t)}, \tag{4.11}$$

and the T_α-forward basis-point-value (also called annuity, see e.g., [11]) is

$$BPV_{\alpha\omega}(t) \equiv \sum_{i=\alpha+1}^{\omega} \theta_{i-1} B_{\alpha i}(t). \tag{4.12}$$

A receiver swaption, instead, gives the right to enter in a swap where the owner receives the fixed rate K_S

$$\begin{aligned}\mathbf{RS}_{\alpha\omega}(T_0) &\equiv E[e^{-\int_{T_0}^{T_\alpha} r_t dt} BPV_{\alpha\omega}(T_\alpha)(K_S - S_{\alpha\omega}(T_\alpha))^+] \\ &= B_{0\alpha}(T_0)E^{(\alpha)}[BPV_{\alpha\omega}(T_\alpha)(K_S - S_{\alpha\omega}(T_\alpha))^+].\end{aligned}$$

As well known in the literature a swaption can be always viewed as a bond option.

Payer and receiver are equivalent to

$$\begin{aligned}\mathbf{PS}_{\alpha\omega}(T_0) &= B_{0\alpha}(T_0)E^{(\alpha)}[1 - P_{\alpha\omega}(K_S; T_\alpha)]^+, \\ \mathbf{RS}_{\alpha\omega}(T_0) &= B_{0\alpha}(T_0)E^{(\alpha)}[P_{\alpha\omega}(K_S; T_\alpha) - 1]^+.\end{aligned} \tag{4.13}$$

and then equal respectively to a put and a call coupon bond option with coupon K_S and strike 1. This relation is a direct consequence of definitions (4.11) and (4.12).

Using the same data of previous subsection, in Fig. 3 we plot a 5y-5y payer swaption according to the exact solution as a function of the strike K_S [see Eqs. (4.13) and (4.8)]. In Fig. 4 we plot the differences with the approximate Black formula (4.10). Even in this case differences are negligible.

In this section we have shown that in the BMM it is possible to price with Black-like formulas caps/floors [see Eq. (4.5)], bond options calls (4.9) and puts (4.10) and swaptions. In particular, using the relation with bond options, payer and receiver swaptions become

$$\begin{aligned}\mathbf{PS}^{(a)}_{\alpha\omega}(T_0) &= B_{0\alpha}(T_0)\big[N\big(-d_2^{(S)}\big) - P_{\alpha\omega}(K_S; T_0)N\big(-d_1^{(S)}\big)\big], \\ \mathbf{RS}^{(a)}_{\alpha\omega}(T_0) &= B_{0\alpha}(T_0)\big[P_{\alpha\omega}(K_S; T_0)N\big(d_1^{(S)}\big) - N\big(d_2^{(S)}\big)\big],\end{aligned} \tag{4.14}$$

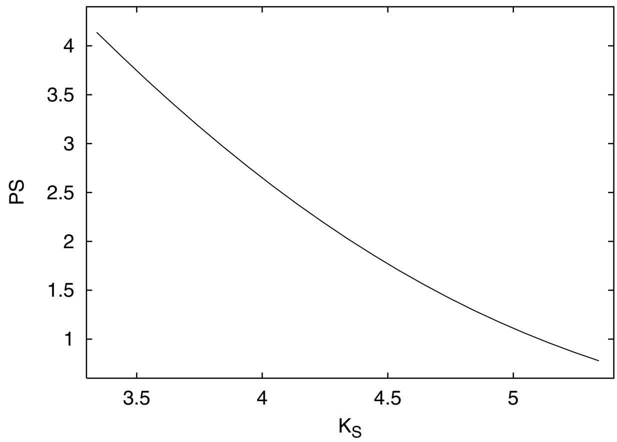

Fig. 3. We show the price of a 5y-5y payer swaption as a function of the strike K_S. The values (in percentage of the outstanding) are obtained via the exact solution [Eqs. (4.11) and (4.8)] with 10^7 random paths.

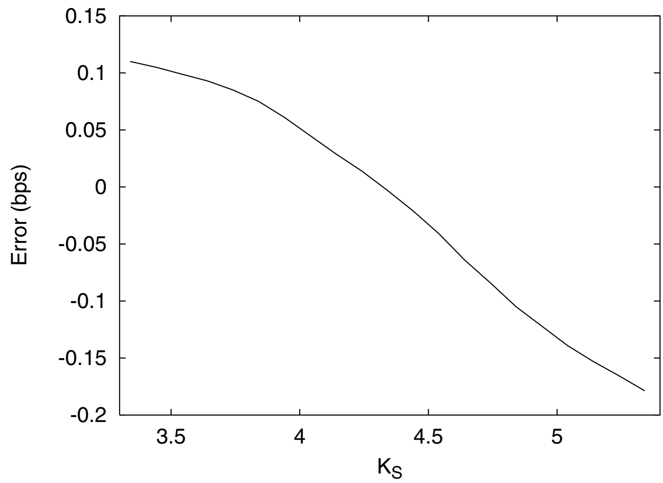

Fig. 4. We plot the differences in bps between the exact solution [obtained through Eq. (4.8)] and the Black approximated formula [via Eq. (4.10)].

where

$$d_1^{(S)} = \frac{1}{\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0}} \ln P_{\alpha\omega}(K_S; T_0) + \frac{1}{2}\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0}\,,$$
$$d_2^{(S)} = \frac{1}{\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0}} \ln P_{\alpha\omega}(K_S; T_0) - \frac{1}{2}\mathcal{V}_{\alpha\omega}\sqrt{T_\alpha - T_0}\,,$$

and

$$\mathcal{V}_{\alpha\omega}^2 = \frac{1}{T_\alpha - T_0}\int_{T_0}^{T_\alpha} \tilde{V}_{\alpha\omega}(K_S;t)\rho\tilde{V}_{\alpha\omega}(K_S;t)dt.$$

Let us notice, from the above equations, that ATM swaptions are equal to ATM bond options.

5. Comparison with Libor Market Model and Swap Market Model

In this section we allow $v_i(t)$ to be a generic (adapted) Markov process and not necessarily a deterministic function of time in order to compare the BMM with the other two market models: the LMM and the SMM. In particular we discuss the relation between price vol and yield vol.

The next proposition deduces the well known results of the LMM [5, 17] and relates the volatility of $B_i(t)$ with the one of the corresponding Libor rate.

Proposition 5.1. *In the* $(i+1)$*-forward measure the dynamics of the forward Libor rate* $L_i(t)$ *becomes*

$$dL_i(t) = -L_i(t)\sigma_i(t)dW^{\{(i+1)\}}(t)\,, \tag{5.1}$$

and the i*th plain vanilla caplet with strike* K *defined in Eq.* (4.3) *is*

$$c_i(T_0) = B_{0i+1}(T_0)E^{(i+1)}[L_i(T_i) - K]^+, \tag{5.2}$$

where we define

$$\sigma_i(t) \equiv \frac{v_i(t)}{\theta_i B_i(t) L_i(t)}. \tag{5.3}$$

Proof. From Eq. (3.3) and using relation (2.1), one gets:

$$dL_i(t) = \frac{v_i(t)}{\theta_i B_i(t)}[\rho v_i(t)dt - dW^{\{(i)\}}(t)].$$

The first part of the proposition is proven using the definition (5.3) and rewriting the above equation in the (T_{i+1})-forward measure. The second part is straightforward from the definition of a caplet (4.3). □

If we choose $\sigma_i(t)$ as a deterministic function of time, then $L_i(t)$ is lognormal in the $(i+1)$-forward measure; we get the Libor Market Model solution for caplets (i.e., the Black formula).

Equation (5.3) is equivalent to affirm that

$$v_i(t) \propto \theta_i L_i(t)\sigma_i(t).$$

BMM and LMM are linked via an instantaneous version of the well known relation between price vol (volatility of the ZC) and yield vol (volatility of the Libor rate) as described in [11]. The former is proportional to the latter multiplied by the duration (that in this case is simply θ_i) and the rate (the forward Libor rate $L_i(t)$).

From a practitioner point of view, it is useful at this point to offer some words in comments.

We have shown that calibrating on caps/floors the BMM [via Eqs. (4.5)] is as easy as in the LMM (via Black formulas). However, calibrating a BMM on swaption prices is much simpler then a LMM due to the almost-exact Black-like swaption formula we have derived in the previous section, approximated formula that is not present in the LMM with the same degree of accuracy.

Moreover, in the LMM, $L_i(t)$ dynamics in the spot measure can be deduced from Eq. (5.1) [5, 6]

$$dL_i(t) = L_i(t)\sigma_i(t)\left[\sum_{j=k(t)+1}^{i} \rho_{ij}^{(L)}\sigma_j(t)\frac{\theta_j L_j(t)}{1+\theta_j L_j(t)}dt - dW(t)\right].$$

It always involves the other Libor rates $L_j(t)$ with fixing in T_j (and $t < T_j < T_i$). A consequence of this inseparability of the Libor dynamics is that almost no solution is available for exotic payoffs where different Libor rates are involved. Furthermore, it can be shown [5] that the dynamics of the Libor rate $L_i(t)$ is lognormal only in the (T_{i+1})-forward measure and then only under this measure it is possible to obtain the exact conditional probability distribution of $L_i(T_\beta)$ given $L_i(T_\alpha)$ with $T_\beta < T_\alpha$.

The BMM allows to have, also under the spot measure, the conditional probability distribution of the ZC B_i (and then of the Libor rates L_i) at a reset date $T_\beta \leq T_i$ given the situation at previous reset date $T_\alpha < T_\beta$: this property (that LMM does not have due to the above equation) is crucial in MonteCarlo simulations since it is possible to limit ZC evolution only to reset dates having no discretization bias.

As in the LMM case, it is also possible to establish a relation between the BMM and the SMM, using the following proposition.

Proposition 5.2. *Under the Basis-Point-Value measure* $\mathcal{Q}^{(\alpha\omega)}$ *the swap rate dynamics is*:

$$dS_{\alpha\omega}(t) = -S_{\alpha\omega}(t)\sigma_{\alpha\omega}(t)dW^{(\alpha\omega)}(t), \tag{5.4}$$

and a payer swaption with strike K_S *and maturity* T_α *as introduced in Eq.* (4.11) *is*

$$\mathbf{PS}_{\alpha\omega}(T_0) = B_{0\alpha}(T_0)BPV_{\alpha\omega}(T_0)E^{(\alpha\omega)}[S_{\alpha\omega}(T_\alpha) - K_S]^+, \tag{5.5}$$

where $E^{(\alpha\omega)}[\cdot]$ *is the* ($\mathcal{F}_0$ *conditional*) *expected value with respect to the measure* $\mathcal{Q}^{(\alpha\omega)}$ *and* $W^{(\alpha\omega)}(t)$ *is a Brownian motion under this measure. We have defined*

$$\sigma_{\alpha\omega}(t) \equiv \frac{V_{\alpha\omega}(S_{\alpha\omega};t)}{BPV_{\alpha\omega}(t)S_{\alpha\omega}(t)}. \tag{5.6}$$

Proof. Using (3.3) in the T_α-forward measure we obtain that the dynamics of the forward swap rate (4.11) is

$$dS_{\alpha\omega}(t) = \frac{V_{\alpha\omega}(S_{\alpha\omega};t)}{BPV_{\alpha\omega}(t)}[\rho\Sigma_{\alpha\omega}(t)dt - dW^{(\alpha)}(t)],$$

and for the basis-point-value (4.12) is

$$dBPV_{\alpha\omega}(t) = BPV_{\alpha\omega}(t)\Sigma_{\alpha\omega}(t)dW^{(\alpha)}(t),$$

with

$$\Sigma_{\alpha\omega}(t) = \sum_{i=\alpha+1}^{\omega} \frac{\theta_{i-1}B_{\alpha i}(t)}{BPV_{\alpha\omega}(t)} v_{\alpha i}(t).$$

Applying the Girsanov transform

$$dW^{(\alpha\omega)}(t) = dW^{(\alpha)}(t) - \rho\Sigma_{\alpha\omega}(t)dt,$$

we obtain (5.4) and the Radon–Nikodym derivative associated is (see, e.g., [8])

$$\frac{d\mathcal{Q}^{(\alpha\omega)}}{d\mathcal{Q}^{(\alpha)}} = \mathrm{Exp}\left\{-\frac{1}{2}\int_{T_0}^{T_\alpha} \Sigma_{\alpha\omega}(t)\rho\Sigma_{\alpha\omega}(t)dt + \int_{T_0}^{T_\alpha} \Sigma_{\alpha\omega}(t)dW^{(\alpha)}(t)\right\},$$

where $W_t^{(\alpha\omega)}$ is a Brownian motion in $\mathcal{Q}^{(\alpha\omega)}$.

Observing that

$$\begin{aligned}\mathbf{PS}_{\alpha\omega}(T_0) &= B_{0\alpha}(T_0)E^{(\alpha)}[BPV_{\alpha\omega}(T_\alpha)(S_{\alpha\omega}(T_\alpha) - K_S)^+] \\ &= B_{0\alpha}(T_0)BPV_{\alpha\omega}(T_0)E^{(\alpha)}\left[\frac{d\mathcal{Q}^{(\alpha\omega)}}{d\mathcal{Q}^{(\alpha)}}(S_{\alpha\omega}(T_\alpha) - K_S)^+\right],\end{aligned}$$

the proposition is proven. □

Equation (5.4) is $S_{\alpha\omega}(t)$ dynamics in the Swap Market Model, i.e., $S_{\alpha\omega}(t)$ is log-normal and martingale in the $\mathcal{Q}^{(\alpha\omega)}$ measure if we choose $\sigma_{\alpha\omega}(t)$ as a deterministic function of time. In this case the swaption price (5.5) is the Black formula first introduced in [18]. It is also useful to notice that relation (5.6) is equivalent to affirm that

$$V_{\alpha\omega}(S_{\alpha\omega};t) = BPV_{\alpha\omega}(t)S_{\alpha\omega}(t)\sigma_{\alpha\omega}(t),$$

then obtaining a generalization of relation (5.3) to the case of swaptions and coupon bond options.

Price vol (volatility of a bond with coupons equal to the swap rate) is proportional (instantaneously) to yield vol (volatility of the swap rate). The proportionality coefficient depends on the yield (the swap rate $S_{\alpha\omega}(t)$) and on the basis-point-value (that is well approximated by the bond duration when the coupon is equal to the at-the-money swap rate). As we have previously mentioned, this relation is commonly used in the market (see, e.g., [11]).

6. Conclusions

In this paper we have described a model of the term structure that allows to price with closed formulas the three classes of OTC plain vanilla interest rate derivatives. The model proposed in this paper, named Bond Market Model, allows to price (almost exactly) coupon bearing bond options, with the same formula used (as described in the introduction) by practitioners in the market, as the Libor Market Model does for caps/floors (options on Libor rates) and the Swap Market Model does for swaptions (options on swaps). Furthermore, within BMM also caps/floors and swaptions are priced with Black-like solutions. This fact allows a calibration of the BMM with a high degree of accuracy on market prices of both the volatilities $v_i(t)$ and the correlations ρ_{ij}.

A comparison with the other two market models is discussed in detail. We summarize the main relations in the following Table 3.

The Swap Market Model allows to price swaptions according to Black; however, it is not known how to manage caps/floors within the model. The Libor Market Model, reproducing Black formula for caps and floors, looks very powerful, even if it is not available for swaptions a closed form solution with the same degree of accuracy of the BMM. We have shown the link between volatilities in the three market models via some relations that are very similar to those used in the market.

From a practitioner point of view, in LMM it is not straightforward to calibrate on market prices the correlation part of the model due to the absence of an easy-to-handle good approximation of swaptions. Moreover, the main theoretical drawback of LMM is that (see, e.g., [13] for a detailed discussion) different Libor models are not compatible; quarterly and semiannual tenor LMMs are inconsistent and both are inconsistent with SMM. Clearly the assumption that quarterly Libor volatility is deterministic is not more compelling than semiannual Libor volatility being deterministic.

The Bond Market Model presents several advantages compared with the other market models, from both a practical and a theoretical perspective. We have discussed how it is possible to price via Black-like formulas the three classes of plain vanilla OTC derivatives; the shown results are valid whatever is the number M of factors considered and whatever is the lag between reset dates, paving the way to a more accurate theoretical description of the term structure dynamics.

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Table 3. A comparison between the three market models.

	Bond Market Model	Libor Market Model	Swap Market Model
Basic Dynamics	$\frac{dB_{\alpha\omega}(t)}{B_{\alpha\omega}(t)} = v_{\alpha\omega}(t)dW^{(\alpha)}(t)$	$\frac{dL_i(t)}{L_i(t)} = -\sigma_i(t)dW^{(i+1)}(t)$	$\frac{dS_{\alpha\omega}(t)}{S_{\alpha\omega}(t)} = -\sigma_{\alpha\omega}(t)dW^{(\alpha\omega)}(t)$
Black Formula	ZC bond: $\mathcal{C}^{(B)}_{\alpha\omega}$, $\mathcal{P}^{(B)}_{\alpha\omega}$ Coupon bond: $\mathcal{C}^{(a)}_{\alpha\omega}$, $\mathcal{P}^{(a)}_{\alpha\omega}$	Cap/Floor: C, F	Swaption: **PS**, **RS**
Volatility Relation		$\sigma_i(t) = \frac{v_i(t)}{\theta_i B_i(t) L_i(t)}$	$\sigma_{\alpha\omega}(t) = \frac{V_{\alpha\omega}(S_{\alpha\omega};t)}{BPV_{\alpha\omega}(t)S_{\alpha\omega}(t)}$

In the first line we report, for each market model, underlying dynamics in the appropriate forward measure: zero coupon $B_{\alpha\omega}$ dynamics in the T_α-forward-measure for BMM, Libor L_i dynamics in the T_{i+1}-forward-measure for LMM, swap rate $S_{\alpha\omega}$ dynamics in the corresponding forward swap measure for SMM.

In the second line we mention, for each market model, the cases where model solutions are equal to Black formulas: bond options for BMM, caps/floors for LMM and payer/receiver swaptions for SMM.

In the last line we show the link between yield and price volatilities via the corresponding yield and duration.

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