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A perturbative approach to Bermudan options pricing with applications

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In this paper we address the problem of the valuation of Bermudan option derivatives in the framework of multi-factor interest rate models. We propose a solution in which the exercise decision entails a properly defined series expansion. The method allows for the fast computation of both a lower and an upper bound for the option price, and a tight control of its accuracy, for a generic Markovian interest rate model. In particular, we show detailed computations in the case of the Bond Market Model. As examples we consider the case of a zero coupon Bermudan option and a coupon bearing Bermudan option; in order to demonstrate the wide applicability of the proposed methodology we also consider the case of a last generation payoff, a Bermudan option on a CMS spread bond.

Keywords: Options pricing; Options applications; Multi-factor models; Bond yields

1. Introduction

The evaluation of Bermudan options in multi-factor interest rate models has always been challenging for practitioners. A Bermudan call (put) bond option is an option that gives the holder the right to buy (sell) at any date t_i (of a set of exercise opportunities $\{t_i\}_{i=1,\dots,N}$) at a strike price K_i a bond with expiry in t_{N+1}, providing that the right has not been exercised at any previous date in the schedule. The difficulty in dealing with such derivatives stems from the determination of the optimal exercise decision.

This paper illustrates a method to evaluate a lower bound for the Bermudan option in a multi-factor interest rate framework via a series expansion in the exercise decision: this allows fast and accurate control of the approximation. Furthermore, in order to demonstrate the quality of the proposed lower bound approximation, an upper bound is provided via the duality technique. We focus on Bermudan bond options, but similar results hold in the swaption case, since the two classes of derivatives are financially equivalent.

In the next section we delineate the Bermudan option pricing problem. In sections 3 and 4 we introduce the technique. In section 5 we briefly describe the Bond Market Model that is used in the numerical examples of sections 6 (zero coupon and coupon bearing) and 7 (CMS spread); there we discuss in detail the accuracy features of the proposed perturbative approach. Finally, in section 8, we summarize the main results and state some concluding remarks.

2. Problem formulation

We briefly introduce the notation in a way similar to Glasserman (2003). We focus our attention on an underlying $\mathbf{B}(t)$ that is an $\Re^d$-valued Markov process. In particular, the jth component of the underlying, $B_j(t)$, can be, for example, the forward price in t of the zero coupon bond that starts in t_j and pays 1 in t_{j+1}.¶ Let us define $h_i(\mathbf{B}(t_i))$ as the payoff received by the option holder by exercising at time t_i, D_{in} as the discounting between two reset dates t_i and t_n, and the continuation value C_i of

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¶Another possible example of $B_j(t)$ can be the jth forward Libor rate in a Libor Market Model.

a Bermudan option at the exercise opportunity t_i as the value of holding rather than exercising the option.

The price C_0 of a Bermudan option in t_0, starting from the initial condition $\mathbf{B}(t_0)=\mathbf{B}^0$, is the value achieved by exercising optimally. From the holder's point of view, the Bermudan option pricing problem can be formulated as the discounted payoff exercising at time t_i with the condition of not having exercised the option previously

$$C_0(\mathbf{B}^0) = \sup_{\tau\in\mathcal{T}} E_0[D_{0\tau}h_\tau(\mathbf{B}(\tau))\,|\,\mathbf{B}(t_0)=\mathbf{B}^0], \quad (1)$$

where $E_0[\cdot]$ is the expectation conditional on the information available at time t_0 under the risk-neutral measure and $\mathcal{T}$ the class of admissible stopping times with values in $\{t_i\}_{i=1,\ldots,N}$.

In order to choose the optimal exercise, the option holder compares at each exercise date, on one hand, the payoff he obtains by exercising the option immediately, and, on the other, the value of holding it and waiting for later exercise dates. The holder's optimal behavior is to exercise in t_i only if

$$h_i(\mathbf{B}(t_i)) \geq C_i(\mathbf{B}(t_i)). \quad (2)$$

The optimal stopping rule is to exercise when condition (2) is satisfied for the first time.

We observe at this point that the continuation value function C_i is itself a Bermudan option with exercise opportunities $t_{i+1}<t_{i+2}<\cdots<t_N$; its value depends only on the underlying $\mathbf{B}$ at time t_i. It is then clear why pricing Bermudan interest rate derivatives within a multi-factor interest rate model is one of the most challenging problems in option pricing theory. In fact, any Monte Carlo valuation of Bermudan options requires, at each exercise opportunity, the continuation value function, i.e. the price of another Bermudan option: an excessive computational effort is needed, as the number of Monte Carlo simulations increases exponentially with the number of exercise opportunities.

For this reason, in practice one estimates lower and upper bounds of the true price. As already underlined several times in the literature (Glasserman 2003 and references therein), the standard technique when looking for a lower bound of a Bermudan option entails two main steps.

- **Step 1:** Find a good suboptimal exercise rule.
- **Step 2:** Compute the expected discounted payoff of the option under this rule.

In order to check the quality of the lower bound, a further step is needed in the evaluation process.

- **Step 3:** Find the associated upper bound using the standard duality technique.

Any approximation $\hat{C}_i(\mathbf{B})$ to the continuation value determines a sub-optimal exercising decision and then a lower bound for the Bermudan option. As stressed by Glasserman (2003, p. 427), 'the option value is usually not very sensitive to the exact position of the exercise boundary—the value is continuous across the boundary—suggesting that even a rough approximation to the boundary should produce a reasonable approximation to the optimal option value'.

In order to implement step 1 in the Bermudan pricing problem and to approximate the boundary of the exercise region at each t_i, the common approach is to choose a parametric class of continuation values $\hat{C}_i(\mathbf{B})$ and then to find the best suboptimal exercise rule within the class. There are two popular methods among practitioners, introduced by Andersen (2000) and Longstaff and Schwartz (2001).

Andersen (2000) defines a threshold at each exercise date t_i and decomposes the maximization of the Bermudan price into $N-1$ subproblems, one for each exercise opportunity except the last. However, it is not easy to maximize the price of a Bermudan option as a function of the exercise boundary, since, on one hand, as we have said, the price value is not very sensitive to the exact location of the exercise boundary, and, on the other, numerical noise is present due to the Monte Carlo estimations of the Bermudan value. Hence, convergence is not granted (Glasserman 2003) and the optimization can be dominated by the noise.

In this paper we propose to substitute the continuation value in the optimal exercise decision (2) with a series expansion of the true C_i; each approximated $\hat{C}_i$ is obtained via a backward induction method starting from the last (non-trivial) continuation value function $\hat{C}_{N-1}$. The method does not involve any maximization.

In their work, Longstaff and Schwartz (2001) approximate the continuation value function, as a function of the underlying state, with a combination of basis functions; the coefficients in this combination are estimated by applying a regression to the simulated paths. The convergence for this method has been proven by allowing the number of simulated paths to increase to infinity while holding the number of basis functions fixed (Clément *et al.* 2002). In practice, for finite simulations, the stability and robustness of the coefficients is often a problematic issue, especially for the typically high-dimensional processes involved in interest rates modeling. Recently, Glasserman and Yu (2004) proved that, in the one-factor log-normal case, in order to ensure the stability of the coefficients, the number of paths in Monte Carlo simulations should increase as $O(\exp(H^2))$, where H is the number of coefficients in the combination of basis functions (these results have recently been generalized by Gerhold (2010)). In spite of the assumed generality of the method, when dealing with multi-dimensional models, coefficient robustness is a very difficult task to achieve even considering an approximated continuation value function that depends only linearly on underlying factors. This limitation of the method is even more binding when computing Greeks.

In this paper we show that when a simple-to-handle function can be utilized as an approximation of the continuation function, there is an elementary way to use this (valuable) information; in the proposed methodology, the coefficients are obtained via a Taylor expansion and are therefore robust.

Furthermore, as we discuss in section 4, the computation of step 3 is much slower than the other two, due to the presence of two nested Monte Carlo simulations. For this reason, an open problem for practitioners has been to try to estimate accuracy by only computing the lower bounds. An interesting recent development in this direction has been suggested by Kolodko and Schoenmakers (2006), who report an iterative construction of lower bounds; unfortunately, even their technique involves nested simulations (in particular, each iteration step requires one more simulation within the simulation).

The perturbative approach we propose in this paper allows an easy estimation of the accuracy achieved via an iterative construction of lower bounds that does not involve nested simulations. This feature has interesting consequences in practice when dealing with outstanding high-dimensional problems such as interest rate Bermudan options pricing.

3. Idea

Let us briefly describe the idea: we approximate the continuation value $\hat{C}_i(\mathbf{B})$ with $\mathcal{M}_i$, a simpler function of $\mathbf{B}$ that has a shape 'reasonably' similar to the continuation value, and then add a correction

$$\hat{C}_i(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + \text{correction}.$$

We note that $C_N = 0$, and that $C_{N-1}(\mathbf{B})$ is a European option that is already a 'simple' function, i.e. $C_{N-1}(\mathbf{B}) = \hat{C}_{N-1}(\mathbf{B}) = \mathcal{M}_{N-1}(\mathbf{B})$. The other corrections are then chosen via an iterative algorithm. Suppose we have already computed all the continuation values $\hat{C}_n(\mathbf{B})$ for $n > i$, we can then calculate $\hat{C}_i(\mathbf{B}^0)$, the Bermudan option in t_i computed on the initial condition by just running one Monte Carlo simulation. This option depends only on the approximated continuation values in the exercise decisions at $t_{i+1}, \ldots, t_{N-1}$. For example, a first approximation of $\hat{C}_i(\mathbf{B})$ can be obtained by choosing

$$\text{correction} = \hat{C}_i(\mathbf{B}^0) - \mathcal{M}_i(\mathbf{B}^0).$$

With this choice on the initial condition $\mathbf{B}^0$ the approximated continuation value is equal to the Bermudan option price $\hat{C}_i(\mathbf{B}^0)$. This is just an example to illustrate the main idea of this paper: we try to approximate the continuation function with a very simple non-parametric rule.

The set of functions $\{\mathcal{M}\}_{i=1,\ldots,N-1}$ we use in this paper is defined through a set of functions $\mathcal{E}_n$. In some cases, these functions can be the European options associated with the Bermudan pricing problem. When $\mathcal{E}_n$ is the European option with exercise date in $t_n > t_i$, valued in t_i on $\mathbf{B}(t_i)$ and with payoff $h_n(\mathbf{B}(t_n))$, $\mathcal{E}_n$ is given by

$$\mathcal{E}_n(\mathbf{B}, t_i) \equiv E_i[D_{in} h_n(\mathbf{B}(t_n)) \,|\, \mathbf{B}(t_i) = \mathbf{B}], \quad i < n \leq N, \quad (3)$$

with $E_i[\cdot]$ the expectation conditional on the information available until time t_i under the risk-neutral measure.

In general, the functions $\mathcal{E}_n$ can be some guessed functions valued in t_i on the initial condition $\mathbf{B}(t_i) = \mathbf{B}$ with a non-null value up to $t_n > t_i$ (after t_n these functions are identically equal to zero). Since the methodology we propose in order to evaluate Bermudan options provides both an upper and a lower bound, we can easily infer the quality of these guess functions. Below, with an abuse of notation, we call the functions $\mathcal{E}_n$ European options even when they are just some guess functions.

In sections 6 and 7 we provide three examples: in the first example, $\mathcal{E}_n$ is a European option, in the second it is an excellent approximation of a European option and in the third example $\mathcal{E}_n$ is just a guess function.

Once we have introduced the functions $\mathcal{E}_n$, we define

$$\mathcal{M}_i(\mathbf{B}) \equiv \mathcal{E}_m(\mathbf{B}, t_i), \quad \text{with } 0 < i < m \leq N, \quad (4)$$

where, among the European options with intermediate expiry $\{\mathcal{E}_n\}_{n=i+1,\ldots,N}$, $\mathcal{E}_m$ is the one with the highest price when valued in t_i on the initial forward discounts $\mathbf{B}^0$

$$\mathcal{E}_m(\mathbf{B}^0, t_i) \geq \mathcal{E}_n(\mathbf{B}^0, t_i), \quad \forall n,\ i < n \leq N.$$

Below we refer to $\{\mathcal{M}_i\}_{i=1,\ldots,N-1}$ as the *maximal* European option valued in t_i.

4. Technique

We show in detail how to construct a sequence of approximated continuation values for the exercise decision (2). The continuation value at the second-last exercise opportunity is $C_{N-1}(\mathbf{B}) = \mathcal{M}_{N-1}(\mathbf{B})$. For the other dates the exact $C_i(\mathbf{B})$ is substituted by $\hat{C}_i^{\{j\}}(\mathbf{B})$ with $j = 0, 1, 2$.

The zeroth-order approximation we consider does not involve any correction term

$$\hat{C}_i^{\{0\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}). \quad (5)$$

The idea is that $\mathcal{M}_i$ is already a reasonable proxy for the true C_i in the exercise decision. The next orders in the approximation are iterative and proceed backwards starting from the continuation value at the last exercise date. When computing the continuation value $\hat{C}_i^{\{j\}}(\mathbf{B})$ at a given approximation level j we assume that we already have all the continuation values $\hat{C}_n^{\{j\}}(\mathbf{B})$ for every $n > i$. The technique is divided into two steps.

- First, we calculate the Bermudan option $\hat{C}_i^{\{j\}}(\mathbf{B}^0)$ on the initial condition $\mathbf{B}^0$, and its Deltas.
- Second, the approximated continuation value $\hat{C}_i^{\{j\}}$ for a generic $\mathbf{B}$ is obtained via the expansion

$$\hat{C}_i^{\{1\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{1\}}(i), \quad (6)$$

$$\hat{C}_i^{\{2\}}(\mathbf{B}) = \mathcal{M}_i(\mathbf{B}) + c_0^{\{2\}}(i) + \sum_{n=i}^{N} c_1^{\{2\}}(i,n)(\ln B_n - \ln B_n^0), \quad (7)$$

where c are constant coefficients. In particular, we choose the coefficients as those obtained in a Taylor expansion†

$$\begin{aligned} c_0^{\{j\}}(i) &= \Big[\hat{C}_i^{\{j\}} - \mathcal{M}_i\Big](\mathbf{B}^0), \quad j = 1, 2, \\ c_1^{\{2\}}(i,n) &= B_n^0\Big[\Delta_n^C - \Delta_n^{\mathcal{M}}\Big](\mathbf{B}^0, i), \end{aligned} \tag{8}$$

where Δ_n is the Delta of the option w.r.t. the zero coupon bond B_n

$$\Delta_n^C(\mathbf{B}^0, i) = \frac{\partial}{\partial B_n^0}\hat{C}_i^{\{2\}}(\mathbf{B}^0),$$
$$\Delta_n^{\mathcal{M}}(\mathbf{B}^0, i) = \frac{\partial}{\partial B_n^0}\mathcal{M}_i(\mathbf{B}^0).$$

We have thus obtained an approximation technique that allows us to compute a sequence of continuation values $\hat{C}_i^{\{j\}}$. At first order, the continuation value is approximated by the function $\mathcal{M}_i$ valued on the particular realization of the discount factor curve in t_i corrected with a (positive) constant $c_0^{\{j\}}(i)$. This constant is equal to the difference between $\hat{C}_i$ and $\mathcal{M}_i$, both valued on the initial discount curve $\mathbf{B}^0$. The assumptions are that $\mathcal{M}_i$ is a good proxy for the true C_i when computing the exercise condition and that, in the region of $\mathbf{B}$ relevant for the valuation, the difference from the true value is approximately constant. This constant is obtained by imposing that the left- and right-hand sides of equation (6) are equal on the initial condition $\mathbf{B}^0$, which is equivalent to considering a zero-order Taylor expansion centered on the initial discount curve. The next approximation level of the true exercise decision is to consider the next term in the Taylor expansion. Therefore, equations (7) and (8) look like a Greeks expansion up to the first order of the Bermudan option. The next term in the expansion will include the Gamma term, and so on. In practice, we take the positive part of the correction, each time the function $\mathcal{M}_i(\mathbf{B})$ is lower than (or equal to) the true continuation value $C_i(\mathbf{B})$, for example when $\{\mathcal{E}_n\}_{n=i,\ldots,N}$ are the 'true' European options associated with the Bermudan option of interest. We stress that this choice of coefficients is quite 'natural' since it includes the financially meaningful terms, namely the Greeks of the option.

Once a proper approximation of the continuation values has been obtained, it is straightforward to compute the expected discounted payoff of the option through a Monte Carlo simulation. In order to verify that this lower bound provides a reasonable estimation of the true value, we have to compute an upper bound. The continuation value function used for the lower bound can also be used for the upper bound through the methodology outlined by Andersen and Broadie (2004) with a technique first proposed by Haugh and Kogan (2004) and Rogers (2001) (see also Jamshidian 2007 for an alternative method). We only recall that the technique involves two nested Monte Carlo simulations. For this reason, for a given numerical precision, the upper bound is much slower to compute than the lower bound of the corresponding Bermudan option.

5. Model

In the numerical example below we limit our attention to the case where the dynamics of the underlying is described by the Bond Market Model, a recently introduced multi-factor interest rate market model (Baviera 2006) belonging to the HJM class that is particularly easy to handle. This model presents several advantages when valuing Bermudan options. In particular, European bond options have Black-like solutions in some simple cases, the discount factor is a deterministic function of the underlying $\mathbf{B}$ and implementing the numerical algorithm is straightforward. In fact, if we consider the fixed set of exercise opportunities $t_1 < t_2 < \cdots < t_N$, the discrete-time process $\mathbf{B}(t_0) = \mathbf{B}^0$, $\mathbf{B}(t_1), \ldots, \mathbf{B}(t_N)$ is a Markov chain on $\Re^d$, and its transition probability can be written explicitly. This allows one to use a large number of factors in a Monte Carlo simulation with little computational effort. In the appendix we briefly recall the main characteristics and some properties of the model. The pricing scheme described above, however, can be used in conjunction with any Markovian model for the interest rates term structure.

6. Numerical results

In order to demonstrate the quality of the approximation, we first provide lower and upper bounds for 20-year Bermudan call options with annual exercise opportunities ($N = 19$) using the same dataset as Baviera (2006) with an evaluation date t_0 of 14 January 2005 at 11:15 a.m. C.E.T.

Figures 1 and 2 summarize the results respectively for a zero coupon (ZC) and for a coupon bearing Bermudan call option in a Bond Market Model with 19 factors for a particular choice of strikes and coupons near the ATM. The underlying of the ZC Bermudan call option is a ZC bond that pays 1 at maturity t_{20} = January 14, 2025; strike K_i is the price that can be paid by the option holder in order to buy the underlying ZC bond at time t_i. The coupon bearing Bermudan call option has as underlying a bond with step-up annual coupons as shown in figure 2. Strike prices are chosen equal to 1 (the notional amount), i.e. the described coupon bearing Bermudan call option is equivalent to a Bermudan swaption on a step-up swap.

At t_0 we value the Bermudan options with a subset of exercise opportunities: by L_α and U_α we denote the lower bound and the upper bound of Bermudan options with the first exercise opportunity at t_α, where $\alpha = 1, \ldots, 19$. The set of exercise dates is then $t_\alpha, t_{\alpha+1}, \ldots, t_N$. In this way we are able to quickly verify the quality of the

†Here we consider an expansion in the logarithm of B_n, since each forward zero coupon bond has a log-normal dynamics in the Bond Market Model. The Libor Market Model case would be similar. This expansion describes the most frequent situation. In a more general case it is possible to consider a Taylor expansion in B_n of the difference between $\hat{C}_i$ and $\mathcal{M}_i$.

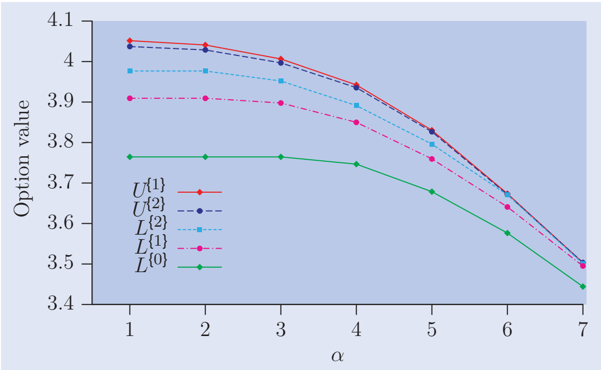

Figure 1. Plot of the annually exercisable zero coupon Bermudan call option values with first exercise date t_α, $\alpha = 1, \ldots, 7$, with $t_1 =$ January 14, 2006, and the last exercise date $t_{19} =$ January 14, 2024. The zero coupon maturity is $t_{20} =$ January 14, 2025. We report lower (L_α) and upper (U_α) bounds with $\alpha = 1, \ldots, 7$ valued in the Euro market on Friday, January 14, 2005, at 11:15 a.m. C.E.T. (t_0). The set of strikes $\{K_i\}_{i=1,\ldots,19}$ is such that $K_i = 8.75 \times 10^{-4} i^2 + 1.09 \times 10^{-2} i + 0.432$. The prices (as a percentage of the notional) according to the Bond Market Model are computed for three orders of approximation for the lower bound $L^{\{j=0,1,2\}}$ and for two orders of approximation for the upper bound $U^{\{j=1,2\}}$. L_α are computed with 10^6 paths and U_α using 5×10^4 paths for the main (outer) simulation and 10^3 paths for the nested (inner) simulation.

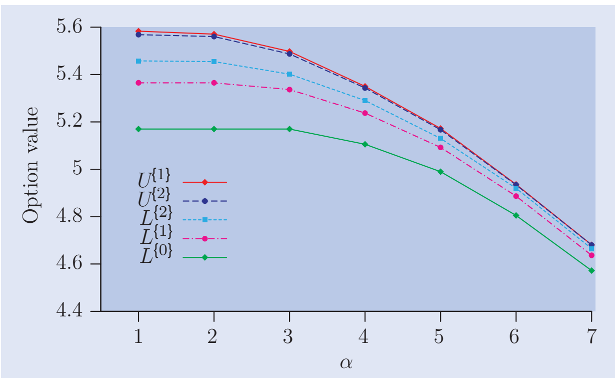

Figure 2. For the same data set as figure 1, we plot annual coupon bond Bermudan call options vs. the first expiry date α ($\alpha = 1, \ldots, 7$). Strikes are equal to 1 and bond annual coupons c_i are stepped up by 0.2% every year for the first 10 years, starting from 2.9% up to 4.7%, and are then constant. Option values (as a percentage of the notional) are shown on the same approximations for lower (L_α) and upper bounds (U_α) and are computed using the same number of paths as in figure 1.

approximation for different numbers of expiries. We plot lower and upper bounds for $\alpha \le 7$; for larger α the different price estimates coincide within the statistical error for the first and second order of the approximation.

Let us stress that we have tested the methodology on a case where Bermudan options are quite far from their

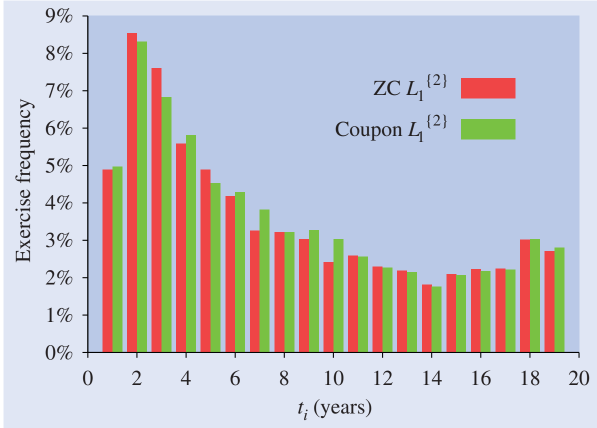

Figure 3. Exercise frequency (in percent) on each exercise date for the Bermudan options on the zero coupon (ZC) and coupon bearing bonds valued in the second-order approximation.

European analog: in fact, as shown in figure 3, the exercise frequency is broadly distributed across all exercise opportunities. We note that even the rough approximation of the lower bound $L_\alpha^{\{0\}}$, where we set the continuation value function equal to the maximal European option, provides reasonable results: prices are approximately 30 bps† lower than the true values for $\alpha = 1$. The order one approximation already provides a very good estimate of the true price, implying that the continuation value function is well approximated by the maximal European option plus constant c_0. The second order provides an excellent estimate of the true value, which lies well within market bid/ask spreads.

We considered several sets of strikes, obtaining similar results. In particular, the method was tested for deeply OTM Bermudan options:‡ for example, in a deeply OTM step-down 20-year Bermudan option§ we observe that even with the 'rough' approximation (the zeroth order) we find excellent results: we obtain a lower bound a couple of bps lower than the mid-price (equal to 77 bps using the same number of paths as figures 1 and 2), well within the market mid-bid spreads. Such a small error, for a long Bermudan option valued with a 19-factor model, is due to the Monte Carlo noise when computing the upper bound and to the approximations discussed in appendix A. In this case the first-order approximation improves the 'rough' lower bound of only half a basis point.

6.1. Accuracy

Let us comment briefly on accuracy. The standard way to evaluate the accuracy achieved is to consider the difference between the upper and the lower bound: $\mathcal{A}_{\rm std}^{\{j\}} = U_\alpha^{\{j\}} - L_\alpha^{\{j\}}$. The perturbative approach described here allows an estimation of the accuracy with just

†A basis point (bp) is 0.01%.
‡We thank an anonymous referee for suggesting this analysis.
§Annual coupons c_i are chosen equal to 7% for the first year, 6% for the next two years, 5% for the next three years, 4% for the next four years, and 1% from the 11th up to the last year.

Table 1. Estimated lower bound price and accuracy for the ZC Bermudan option.

α	$L_{\alpha}^{\{2\}}$ (%)	$\mathcal{A}_{\text{std}}^{\{2\}}$ (bp)	$\mathcal{A}_{\text{est}}^{\{2\}}$ (bp)
1	3.977	6	6
2	3.977	5	6
3	3.952	4	5
4	3.891	4	4
5	3.796	3	4
6	3.675	1	2
7	3.516	0	2

lower bounds. In this case, the jth order accuracy $\mathcal{A}_{\text{est}}^{\{j\}}$ can be obtained by computing

$$\mathcal{A}_{\text{est}}^{\{j\}} \equiv L_{\alpha}^{\{j\}} - L_{\alpha}^{\{j-1\}}. \quad (9)$$

Table 1 shows, for the ZC case, the standard accuracy and the estimation proposed. For example, for $\alpha = 1$, the estimation of the accuracy with just lower bounds (9) is approximately 6 bps, i.e. very similar to the accuracy obtained from the upper bound. As stressed above, the computation of upper bounds, involving two nested Monte Carlo simulations, requires longer computational times than the corresponding lower bounds. When one needs a fast estimation of the accuracy achieved, $\mathcal{A}_{\text{est}}$ is an excellent solution since it is of the same order of magnitude of the true accuracy.

7. Callable CMS spread

To illustrate the versatility of the proposed methodology, we now apply it to the pricing of Bermudan options struck on an underlying bond that pays at each reset date the difference between two CMS rates with different tenor (CMS spread). The bond is callable at par. This kind of option is embedded in a large set of structured callable bonds that have been issued in the last few years. In 2005 the issuance of leveraged CMS spread bonds in the EURO market amounted to almost EUR 10 billion (Citigroup 2006) and a significant part of these have callable features. These bonds pay coupons that are proportional to the CMS spread, and coupons are often both floored and capped. The ith coupon is therefore

$$c_i = \min\{\max[z(S_{i;\tau_1}(t_i) - S_{i;\tau_2}(t_i)), K_i^L], K_i^U\},$$

where K_i^L and K_i^U are the floor and the cap strike respectively, z is the leverage, the CMS rates $\{S_{i;\tau}\}$ with tenor τ are fixed at t_i and the coupon is paid at t_{i+1} (fixed in advance, payment in arrears). For this kind of payoff, where different rates are compared at each expiry, it is crucial to use a multi-factor interest rate model for the curve's dynamics. As pointed out above, multi-factor models make the use of standard pricing schemes of Bermudan options extremely time consuming.

Even if an analytical expression for the European option on a collared CMS spread bond is not available, we can introduce a set of functions $\mathcal{M}_i$ that can be used in the expansion of the continuation values (see appendix B for further details). This example shows that, in the proposed computational scheme, it is not necessary for $\mathcal{E}_n$ to be a European option, but it is enough to have a good guess that is fast to compute.

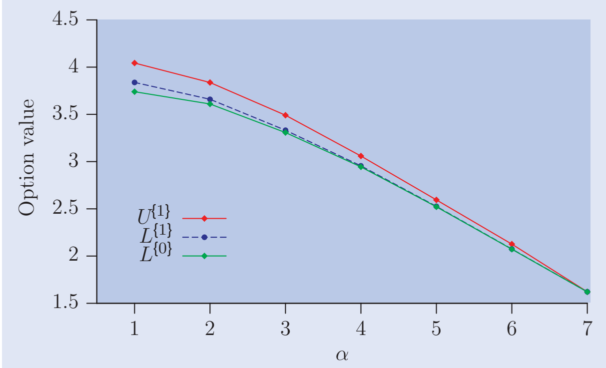

Figure 4. For the same data set as figure 1, we plot annual 10 – 2 CMS spread Bermudan call options vs. the first expiry date α ($\alpha = 1, \ldots, 7$). Option values (as a percentage of the notional) are shown for approximation 1 for lower (L_α) and upper bounds (U_α) and are computed using the same number of paths as in figure 1.

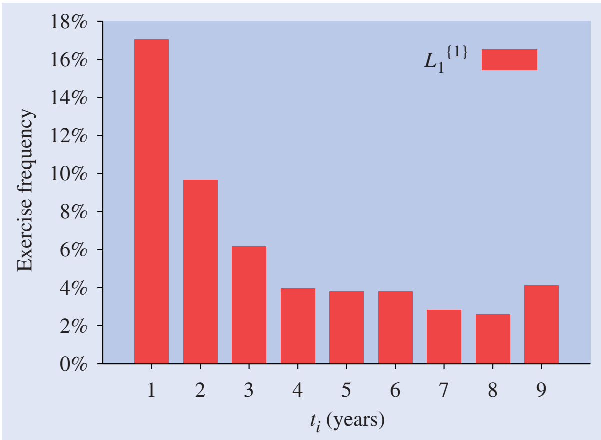

Figure 5. Exercise frequency (in percent) on each exercise date for the Bermudan option on a collared 10 – 2 CMS spread bond valued in the first-order approximation.

In figure 4 we consider a 10-year call option with annual exercise opportunities ($N = 9$) on a bond where $\tau_1 = 10$, $\tau_2 = 2$, $z = 5$, $K^L = 0.5\%$ and $K^U = 8\%$. In order to value this callable option we use a Bond Market Model with 19 factors, as in previous sections. As before, the evaluation time is 14 January 2005 at 11:15 a.m. C.E.T. We consider the methodology only up to the first order of approximation since both the lower and upper bounds are already well within the market bid/ask. In figure 5 we show the exercise frequency of the Bermudan option across the exercise opportunities.

8. Conclusions

We have devised a computational scheme to price Bermudan call options that is superior to standard approaches, which can be time consuming when noisy quantities are involved (as is the case when dealing with expected values obtained through Monte Carlo simulations). The methodology is based on a Taylor series expansion of the exercise condition, which is expressed in terms of financially meaningful quantities (option's Greeks). The basic ingredient is a simple-to-compute approximation of the continuation function and a Monte Carlo simulation mechanism for the underlying interest rate dynamics. We stress that the methodology is quite general for pricing callable products, since we do not need a closed form for the corresponding European options, but a simple guess in order to approximate the continuation function: even some payoffs that are path-dependent and with a relevant digital risk can be treated within the described methodology. In the examples discussed, the presence either of Black-like solutions for the European options or of reasonable proxy functions allow for a fast computation of the price.

In the plain vanilla cases, the second-order approximation is already very precise: the accuracy is less than a few basis points for all expiries considered. This result is confirmed irrespective of the choice of option moneyness. In the case where we consider a CMS spread, already with the first-order approximation we obtain an accuracy well within the market bid/ask. Moreover, if even greater accuracy is required, one can always consider the next steps in the perturbative approach.

We have thus obtained a fast and accurate algorithm for the valuation of Bermudan options. Speed and accuracy are of crucial importance when managing exotic options, especially if the computation of Greeks is also required.

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Appendix A: Bond Market Model

In this appendix we briefly recall the Bond Market Model, a multi-factor Heath–Jarrow–Morton model (Heath *et al.* 1992), and report the analytical formulas for pricing European bond options. A detailed description of the model and the main results can be found in Baviera (2006). In the spot measure, the dynamics for each (forward) zero coupon B_j is

$$dB_j(t) = B_j(t)v_j(t)\left[-\sum_{l=k(t)+1}^{j-1} \rho v_l(t)dt + dW(t)\right], \qquad (A1)$$

where $v_j(t)$, $j = 1, \ldots, N$, are d-dimensional vectors of deterministic functions of time, with $v_j(t) = 0$ for $t \geq t_j$, W is a d-dimensional Brownian motion with instantaneous covariance $\rho = (\rho_{lm=1,\ldots,d})$, and the function $k(t)$: $\Re \to \mathbb{N}$ denotes the previous reset date

$$k(t) = k, \quad \text{when } t_k \leq t < t_{k+1}.$$

The three main properties of the Bond Market Model when pricing Bermudan options are:

(1) the discrete-time process $\mathbf{B}(t_0) = \mathbf{B}^0, \mathbf{B}(t_1), \ldots, \mathbf{B}(t_N)$ is a Markov chain whose transition probability can be written explicitly and the dynamics is trivial to simulate;
(2) the discount factor can be written as a function of $\mathbf{B}$

$$D_{in} = \prod_{j=i}^{n-1} B_j(t_j);$$

(3) call and put European options on both zero coupon and coupon bearing bonds can be written as Black formulae.

For a coupon bearing Bermudan option with strike K_i, the payoff at t_i is

$$h_i(\mathbf{B}(t_i)) \equiv [P_{i\,N+1}(\mathbf{B}(t_i);\mathbf{c}) - K_i]^+,$$

where the superscript $^+$ denotes the positive part of the argument, and the coupon bond that starts at t_i ends at t_{N+1} and has $N+1-i$ calculation periods with coupon vector $\mathbf{c} = \{c_r\}_{r=i,\ldots,N}$,

$$P_{i\,N+1}(\mathbf{B}(t_i);\mathbf{c}) \equiv \sum_{r=i}^{N} c_r\theta_r B_{i\,r+1}(t_i) + B_{i\,N+1}(t_i),$$

with lag $\theta_i \equiv t_{i+1} - t_i$. The forward zero coupon bond that starts at t_i and pays 1 at t_{N+1} is

$$B_{i\,N+1}(t) \equiv \prod_{n=i}^{N} B_n(t), \quad \text{for } i \le N.$$

The price of a call option at t_i on the coupon bearing bond $P_{nN+1}(\mathbf{B};\mathbf{c})$ with exercise date $t_n > t_i$ and strike K_n is a Black-like formula

$$\mathcal{E}_n(\mathbf{B}, t_i) = B_{sn}\{P_{nN+1}(\mathbf{B};\mathbf{c})\mathcal{N}[d_1^{(n)}(\mathbf{B}, t_i)] - K_n\mathcal{N}[d_2^{(n)}(\mathbf{B}, t_i)]\}, \tag{A2}$$

where

$$d_1^{(n)}(\mathbf{B}, t_i) = \frac{1}{\mathcal{V}_{nN+1}\sqrt{t_n - t_i}}\ln\frac{P_{nN+1}(\mathbf{B};\mathbf{c})}{K_n} + \frac{1}{2}\mathcal{V}_{nN+1}\sqrt{t_n - t_i},$$
$$d_2^{(n)}(\mathbf{B}, t_i) = \frac{1}{\mathcal{V}_{nN+1}\sqrt{t_n - t_i}}\ln\frac{P_{nN+1}(\mathbf{B};\mathbf{c})}{K_n} - \frac{1}{2}\mathcal{V}_{nN+1}\sqrt{t_n - t_i}$$

and

$$\mathcal{V}^2_{nN+1} = \frac{1}{t_n - t_i}\sum_{i,j=n+1}^{N+1}\gamma_i^{nN+1}\gamma_j^{nN+1}\sum_{l=n}^{i-1}\sum_{m=n}^{j-1}\int_{t_i}^{t_n} v_l(t)\rho_{lm}v_m(t)\mathrm{d}t,$$

where we define

$$\gamma_i^{nN+1} \equiv \frac{1}{\sum_{r=n+1}^{N+1} c_{r-1}\theta_{r-1}B^0_{nr} + B^0_{nN+1}} \cdot \begin{cases} c_{i-1}\theta_{i-1}B^0_{ni}, & i \le N, \\ (1 + c_N\theta_N)B^0_{nN+1}, & i = N+1. \end{cases}$$

The zero coupon European call option is the limit for zero coupons of equation (A2).

Appendix B: Collared CMS spread

In this appendix we show that, within the Bond Market Model, a suitable set of functions $\mathcal{E}_n(\mathbf{B}, t_i)$ can also be defined in a last generation payoff as a collared CMS spread Bermudan option. When a payoff of a curve's spread is involved, we clearly need a multi-factor interest rate model. In order to apply the proposed methodology we need to specify only the set of functions $\mathcal{E}_n(\mathbf{B}, t_i)$. From these, as explained in section 2, one can easily deduce the set of functions $\mathcal{M}_i$ that are the basic ingredients in the expansion.

Three sets of dates (and the corresponding indices) are relevant.

- t_i: the date when the Bermudan option holder has the right to exercise the option, i.e. the date where we need to value the function $\mathcal{M}_i$ in the proposed approximation procedure;
- t_n: the expiry date of each function $\mathcal{E}_n(\mathbf{B}, t_i)$ valued at t_n ($i < n \le N$);
- t_j: the fixing date of each coupon in the underlying bond of $\mathcal{E}_n(\mathbf{B}, t_i)$ ($n \le j \le N$).

The two CMS swap rates in each coupon (for simplicity, we assume that the jth coupon is fixed in advance at t_j and is paid annually with a 30/360 day count convention at t_{j+1}) have respectively the tenor τ_1 and τ_2. The coupon is floored by K_L and capped by K_U, the leverage is z and k_i is the issuer's funding spread over Libor for the period (t_i, t_{i+1}). This funding spread is assumed to be equal to zero in the numerical example reported in this paper.

Let us first introduce the basic notation for this payoff. We define the (forward) swap rate at time t fixed at t_j with tenor τ,

$$S_{j;\tau}(t) = \frac{1 - B_{jj+\tau}(t)}{BPV_{jj+\tau}(t)}, \quad \text{with } t \le t_j,$$

the (forward) Libor rate at time t fixed at t_j and calculation period (t_j, t_{j+1}),

$$L_j(t) = S_{j;1}(t) = \frac{1}{\theta_j}\left(\frac{1}{B_j(t)} - 1\right), \quad \text{with } t \le t_j,$$

and the (forward) basis point value

$$BPV_{jm}(t) \equiv \sum_{l=j}^{m-1}\theta_l B_{jl+1}(t), \quad \text{with } t \le t_j.$$

We also define the two sets of weights that are used

$$w_j^n \equiv \frac{B_{nj+1}(t_i)\theta_j}{BPV_{nN+1}(t_i)}, \quad n \le j \le N,$$

and

$$\gamma_k^{jj+\tau} \equiv \begin{cases} S_{j;\tau}(t_0)\theta_{k-1}B_{jk}(t_0), & j < k < j+\tau, \\ (1 + S_{j;\tau}(t_0)\theta_{j+\tau-1})B_{jj+\tau}(t_0), & k = j+\tau. \end{cases}$$

The functions $\mathcal{E}_n(\mathbf{B}(t_i), t_i)$ can be chosen as

$$\mathcal{E}_n(\mathbf{B}(t_i), t_i) = B_{in}(t_i)BPV_{nN+1}(t_i) \times \left\{[\tilde{S}(t_i) - K(t_i)]N(\tilde{d}) + \mathcal{V}\sqrt{\frac{t_n - t_i}{2\pi}}\exp\left(-\frac{\tilde{d}^2}{2}\right)\right\}, \tag{B1}$$

with

$$\tilde{d} = \frac{\tilde{S}(t_i) - K(t_i)}{\mathcal{V}\sqrt{t_n - t_i}}.$$

The volatility $\mathcal{V}$ is defined as

$$\mathcal{V}^2 = \frac{1}{t_n - t_i}\int_{t_i}^{t_n}\tilde{V}(t)\rho\tilde{V}(t)\mathrm{d}t,$$

where

$$\tilde{V}(t) = \sum_{j=n}^{N} w_j^n \left[z(\tilde{V}_{j;\tau_1}(t) - \tilde{V}_{j;\tau_2}(t)) - \tilde{v}_j(t) \right].$$

The effective strike at time t_i is approximated by

$$K(t_i) = \sum_{j=n}^{N} w_j^n k_j,$$

and the effective rate by

$$\tilde{S}(t_i) = \sum_{j=n}^{N} w_j^n \left[z \min[\max[\tilde{S}_{j;\tau_1}(t_i) - \tilde{S}_{j;\tau_2}(t_i), K_L/z], K_U/z] - \tilde{L}_j(t_i) \right].$$

The CMS rates $\tilde{S}_{j;\tau}(t_i)$ and the Libor rate $\tilde{L}_j(t_i)$ in the above formula can be seen as the swap and Libor rates plus corrections (that are specified below)

$$\tilde{S}_{j;\tau}(t_i) = S_{j;\tau}(t_i) + \delta S_{j;\tau}(t_i),$$
$$\tilde{L}_j(t_i) = L_j(t_i) + \delta L_j(t_i)$$

and rate volatilities are

$$\tilde{V}_{j;\tau}(t) \equiv \frac{1}{BPV_{j\,j+\tau}(t_0)} \sum_{k=j+1}^{j+\tau} \gamma_k^{j\,j+\tau} v_{jk}(t),$$
$$\tilde{v}_j(t) \equiv \frac{v_j(t)}{B_j(t_0)\theta_j} \mathbf{e}_j,$$
$$v_{jk}(t) \equiv \sum_{l=j}^{k-1} v_l(t) \mathbf{e}_l,$$

for $t \leq t_j$ (otherwise they are equal to zero); $\{\mathbf{e}_l\}_{l=1,\ldots,d}$ is the set of base vectors in $\Re^d$. In the Bond Market Model, rate corrections can be approximated in t_i as

$$\begin{aligned}\delta\tilde{S}_{j;\tau}(t_i) = &(\mathcal{V}_{j;\tau}\rho\Sigma_{j;\tau} - \mathcal{V}_{j;\tau}\rho v_j)(t_j - t_n) \\ &+ (\mathcal{V}_{j;\tau}\rho\Sigma_{j;\tau} - \mathcal{V}_{j;\tau}\rho\Sigma_{n\,N+1} + \mathcal{V}_{j;\tau}\rho v_{nj})(t_n - t_i) \\ &+ b_j(t_i),\end{aligned}$$
$$\delta\tilde{L}_j(t_i) = (-\tilde{v}_j\rho\Sigma_{n\,N+1} + \tilde{v}_j\rho v_{n\,j+1})(t_n - t_i),$$

where

$$\mathcal{V}_{j;\tau}\rho\Sigma_{j;\tau} \equiv \frac{1}{t_j - t_n} \int_{t_n}^{t_j} \tilde{V}_{j;\tau}(t)\rho\tilde{\Sigma}_{j;\tau}(t)\mathrm{d}t,$$
$$\mathcal{V}_{j;\tau}\rho\Sigma_{n\,N+1} \equiv \frac{1}{t_j - t_n} \int_{t_n}^{t_j} \tilde{V}_{j;\tau}(t)\rho\tilde{\Sigma}_{n\,N+1}(t)\mathrm{d}t,$$
$$\mathcal{V}_{j;\tau}\rho v_{j;\tau} \equiv \frac{1}{t_j - t_n} \int_{t_n}^{t_j} \tilde{V}_{j;\tau}(t)\rho v_{j;\tau}(t)\mathrm{d}t,$$
$$\tilde{\Sigma}_{j;\tau}(t) \equiv \sum_{k=j}^{j+\tau-1} \frac{\theta_j B_{j\,k+1}(t)}{BPV_{j\,j+\tau}(t)} v_{jk}(t),$$

and $b_j(t_i)$ is a basis correction for the CMS rate $\tilde{S}_{j;\tau}(t)$, a correction that is simply set equal to zero in the numerical examples reported here.